1. \( L = \{ w \in \{a, b\}^* \mid n_a(w) = n_b(w) \} \).

\[
S \rightarrow SS \mid aSb \mid bSa \mid \epsilon
\]

2. \( L = \{ w \in \{a, b\}^* \mid n_a(w) > n_b(w) \} \).

\[
S_0 \rightarrow AS \mid SAS \mid SA
\]

\[
S \rightarrow SS \mid SAS \mid aSb \mid bSa \mid \epsilon
\]

\[
A \rightarrow aA \mid a
\]

**Proof:** Note that any string generated by the above rules has more \( a \)'s than \( b \)'s. We next proceed to show that any string \( w \in L \) can be generated by these rules. We first note that any string \( z \) such that \( n_a(z) = n_b(z) \) must be split into substrings such that \( z = z_1z_2\cdots z_l \) where (i) each \( z_j \) has equal number of \( a \)'s and \( b \)'s, (ii) the first and the last symbols of \( z_j \) are different, and (iii) any such \( z_j \) does not contain a substring that has the same number of \( a \)'s and \( b \)'s but the first and the last symbols are same. For example, \( aabbab \) cannot be such a \( z_j \) since it contains \( abba \), but \( aababb \) can be such a \( z_j \). It is then noted that for any \( w \in L \), \( w \) can be denoted as:

\[
w = a^{l_0}z_1a^{l_1}z_2a^{l_2}\cdots z_ka^{l_k},
\]

where (1) each \( z_i \) satisfies the above three conditions (i) - (iii); (2) for each \( i \), \( 0 \leq i \leq k \), \( l_i \geq 0 \); and (3) \( l_0 + l_1 + \cdots + l_k > 0 \). For example, \( w = aaababbbaaaabbbaaa \) may be decomposed into \( w = aa \cdot ab \cdot ab \cdot ba \cdot a \cdot aabb \cdot aaaa \), where \( l_0 = 2 \), \( z_1 = ab \), \( l_1 = 0 \), \( z_2 = ab \), \( l_2 = 0 \), \( z_3 = ba \), \( l_3 = 1 \), \( z_4 = aabb \), and \( l_4 = 3 \).

From the start state \( S_0 \), one of the following three cases occurs: If \( l_0 > 0 \), \( S_0 \Rightarrow AS \); else if \( l_k > 0 \), \( S_0 \Rightarrow SA \); otherwise, \( S_0 \Rightarrow SAS \). We then recursively apply \( S \rightarrow SS \) or \( S \rightarrow SAS \) such that a single \( S \) generates a substring \( z_j \) satisfying conditions (i)-(iii) above.

Consider the example above: \( w = aaabbbbaaaabbbaaa \). \( w \) is then split into \( a^2z_1z_2z_3a^1z_4a^3 \), and is generated as follows.

\[
\begin{align*}
S_0 & \Rightarrow AS \\
AS & \Rightarrow SS \\
ASS & \Rightarrow SS \\
ASSS & \Rightarrow SAS \\
ASSSAS & \Rightarrow ASSSAS \\
SAS & \Rightarrow AS \\
ASSSAS & \Rightarrow ASSSASA \\
ASSSASA & \Rightarrow aa \cdot z_1z_2z_3az_4aaa, \text{ which is } aaabbbbaaaabbbaaa.
\end{align*}
\]

**Note:** The following rules suggested by Ben in class work correctly. You can verify the correctness using the similar arguments.
$$
S \rightarrow RaR \mid aRR \mid RRa$

$$
R \rightarrow RaR \mid aRR \mid RRa \mid aRb \mid bRa \mid \epsilon$

3. $L = \{w \in \{a, b\}^* \mid n_a(w) \neq n_b(w)\}$.

Note that $L = \{w \in \{a, b\}^* \mid n_a(w) > n_b(w) \text{ or } n_a(w) < n_b(w)\}$; hence, CFG in #2 can be applied.

4. $L = \{w \in \{a, b, c\}^* \mid n_a(w) + n_b(w) = n_c(w)\}$.

$$
S \rightarrow SS \mid aSc \mid cSa \mid bSc \mid cSb \mid \epsilon$

5. $L = \{w \in \{a, b, c\}^* \mid n_a(w) + n_b(w) > n_c(w)\}$.

$$
S_0 \rightarrow TS \mid STS \mid ST$

$$
S \rightarrow SS \mid STS \mid aSc \mid cSa \mid bSc \mid cSb \mid \epsilon$

$$
T \rightarrow aT \mid bT \mid a \mid b$

6. $L = \{w \in \{a, b, c\}^* \mid n_a(w) + n_b(w) \neq n_c(w)\}$.

Note that $L = \{w \in \{a, b, c\}^* \mid n_a(w) + n_b(w) > n_c(w) \text{ or } n_a(w) + n_b(w) < n_c(w)\}$; hence, CFG in #5 can be applied.