Transmission Schedules for
Hypercube Interconnection in
WDM Optical Passive Star Networks *

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Abstract

This paper is concerned with optimal transmission schedules for a hypercube interconnection using WDM (wavelength division multiplexed) optical passive star network. Our model of the network consists of a set $V$ of $N$ nodes with $N$ tunable transmitters and $N$ fixed-tuned receivers (where each node is assigned a transmitter-receiver pair), $k$ ($= 2^b$ for an integer $b$) wavelengths (each wavelength is shared by $N/k$ receivers where $N/k$ is an integer), and tuning time $\delta > 0$ (in units of time slot, for transmitters to tune from one wavelength to another). We assume that $N$ is equal to the number of nodes in the hypercube, and that at any given time slot at most one transmission can be done per wavelength. An optimal transmission schedule for a hypercube interconnection is defined to be the one that schedules transmissions such that for each node $v$ of the hypercube, $v$ transmits once to each of its neighboring nodes within a repeating cycle of minimum length. In this paper, we first show that the $n$-dimensional hypercube $Q_n$ can be embedded in $V$ such that for any node of $Q_n$, its $n$ neighboring nodes are evenly partitioned into $b$ groups where each group shares a wavelength. Under this embedding, we then present an optimal transmission schedule for each tuning time.
1 Introduction

The interconnection network in a multiprocessing system is of major importance to the performance of the system. Massively parallel processing requires massively parallel interconnects. It is well understood that electronic interconnect faces its fundamental physical limits as the performance of processors and their affiliated memories grow. Optical interconnects (primarily because of their higher bandwidth advantages) are viable alternatives for the traditional electronic interconnects when designing massively parallel processing systems [3, 7, 11, 12]. Interconnection network in a system must support flexible application-dependent virtual topologies. Recently, there has been a lot of interest in using the wavelength division multiplexed (WDM) optical communication network as the underlying architecture to support many different communication patterns by embedding them directly into the system hardware [1, 4, 6, 10, 13, 14, 15].

In this paper, we consider the following WDM optical passive star network [2, 5, 6]. The physical architecture of the network has \( N \) inputs and \( N \) outputs connected by an optical passive star. Figure 1 shows \( N \) nodes connected via an optical passive star. Each node \( v_i \) \((0 \leq i \leq N - 1)\) in the network consists of a pair of transmitter \( t_i \) and receiver \( r_i \). Transmitters and receivers are tuned to any one of the available wavelengths. In order to communicate from a transmitter to a receiver, the corresponding transmitter and receiver must be tuned to the same wavelength. Transmitters and receivers are called tunable if they can tune from one wavelength to another, while ones that cannot are called fixed-tuned.

Our model of the network assumes that receivers are fixed-tuned and transmitters are tunable; it also assumes that there are \( k \) \((2 \leq k \leq N)\) wavelengths in the network and \( N/k \) receivers share a wavelength, where \( N/k \) is an integer for simplicity. Further, we assume that at any given time slot, at most one transmission can be done per
wavelength. Hence, at most \( k \) transmissions can be done simultaneously at any time. Time domain is divided into time slots of equal duration with the slot length equal to the packet duration (i.e., the amount of time to transmit a fixed size packet). For transmitters to tune from one wavelength to another, tuning time \( \delta > 0 \) (expressed in units of time slot) is required, and idle transmitters (i.e., those not involved in transmitting or tuning) can tune to wavelengths just-in-time to start their transmissions. (It is noted that if packet sizes are small such as ATM cells, \( \delta \) is expected to be large.) The problem considered in this paper is formulated as follows:

**Optimal Transmission Schedule Problem:** A WDM optical passive star network consists of a set \( V = \{v_i \mid 0 \leq i < N\} \) of \( N \) nodes with \( N \) tunable transmitters and \( N \) fixed-tuned receivers (where each node \( v_i \) is assigned a transmitter-receiver pair \( (t_i, r_i) \)), \( k \) wavelengths such that each wavelength \( w_i \) for \( 0 \leq i \leq k - 1 \) is shared by \( N/k \) receivers where \( N/k \) is an integer (the set of receivers tuned to wavelength \( w_i \) is \( R_i = \{r_j \mid 0 \leq j < N, j \text{ mod } k = i\} \)), and tuning time \( \delta \). Given a virtual interconnection topology \( G \) with \( N \) nodes that is embedded in the network, the problem is to schedule transmissions in such a way that the time slots are arranged into repeating cycles of minimum length and each node in \( G \) transmits once to each of its neighboring nodes within a cycle (node \( v \) transmits to node \( u \) means that the transmitter assigned to \( v \) transmits to the receiver assigned to \( u \)). In this paper, \( G \) is assumed to be an \( n \)-dimensional hypercube \( Q_n \) (so, \( N = 2^n \)). Thus, an optimal transmission schedule for our model is the one that schedules transmissions such that for each node \( v \) in \( V \), \( v \) transmits once to each of its \( n \) neighboring nodes within a repeating cycle of minimum length.
In this paper, we first construct an embedding $f$ of $Q_n$ in $V$ such that for any node of $Q_n$, its $n$ neighboring nodes are evenly partitioned into $\log_2 k$ groups where each group shares a wavelength. Under the embedding $f$, we then present an optimal transmission schedule with cycle length $\max\{nN/k, n + (\log_2 k)\delta\}$ for each tuning time $\delta$.

In a previous work [8], it is shown that there exists an algorithm for optimal transmission schedules with cycle length $\max\{\frac{N(N-1)}{k}, k\delta + N - 1\}$ when the interconnection topology is $K_N$ (a complete graph with $N$ nodes). Since $Q_n$ is a subgraph of $K_N$ ($N = 2^n$), we see that there exists a transmission schedule for $Q_n$ with cycle length $\max\{\frac{N(N-1)}{k}, k\delta + N - 1\}$. However, it should be remarked that this cycle length is not minimum since $\max\{nN/k, n + (\log_2 k)\delta\} < \max\{\frac{N(N-1)}{k}, k\delta + N - 1\}$.

The following notation will be used throughout this paper. We denote $b = \log_2 k$, $\alpha = n \mod b$, $\beta = b - \alpha$ and $c = \frac{N}{k} \mod b$. For a node $v$ in $Q_n$, $A(Q_n : v)$ denotes the set of $n$ neighboring nodes of $v$ in $Q_n$.

It should be noted that the result obtained in this paper is an improvement on our previous work [9]. In [9], we presented a transmission schedule for each tuning time $\delta$ with cycle length $\frac{nN}{k}$ if $\delta \leq \lfloor \frac{n}{b} \rfloor \left( \frac{N}{k} - 1 \right)$, $n + b\delta$ if $\delta \geq \lceil \frac{n}{b} \rceil \left( \frac{N}{k} - 1 \right)$, and

$$\alpha \left\lfloor \frac{n}{b} \right\rfloor + \beta \left( \left\lceil \frac{n}{b} \right\rceil + \delta \right)$$

if $\left\lfloor \frac{n}{b} \right\rfloor \left( \frac{N}{k} - 1 \right) < \delta < \left\lceil \frac{n}{b} \right\rceil \left( \frac{N}{k} - 1 \right)$.

The rest of this paper is organized as follows. In Section 2, we construct an embedding of $Q_n$ in $V$ such that for any node of $Q_n$, its $n$ neighboring nodes are evenly partitioned into $b$ groups where all the receivers in each group share a wavelength. Based on the embedding in Section 2, Section 3 presents an optimal transmission schedule for each tuning time $\delta$ whose cycle length is $\max\{\frac{nN}{k}, n + b\delta\}$.
2 Embedding

In this section, we construct an embedding of $Q_n$ in $V$ that assigns a wavelength to the receiver of each hypercube node. Our embedding is such that for any node of $Q_n$, its $n$ neighboring nodes are partitioned into $b$ groups (all receivers in each group share a wavelength), where $\alpha$ groups are of size $\left\lfloor \frac{n}{b} \right\rfloor$ and $\beta$ groups are of size $\left\lceil \frac{n}{b} \right\rceil$.

We write $V = \{0, 1, \cdots, 2^n - 1\}$. Each node $v$ of $Q_n$ will be represented by an $n$-bit binary number $b_{n-1}b_{n-2}\cdots b_1b_0$ and $D(v)$ denotes its decimal equivalent. $S_d$ ($b \leq d \leq n$) denotes the $d$-dimensional subcube of $Q_n$ with node set $\{\text{node } v \text{ of } Q_n | 0 \leq D(v) < 2^d\}$. For any two adjacent nodes $u = b_{n-1}b_{n-2}\cdots b_1b_0$ and $v = c_{n-1}c_{n-2}\cdots c_1c_0$ in $Q_n$, dimension of the link $(u, v)$ (i.e., the unique $j$ such that $b_j \neq c_j$) will be denoted by $\text{dim}(u, v)$. For each node $v \in S_d - S_{d-1}$ ($b < d \leq n$), $v(0)$ denotes the unique node in $S_{d-1}$ adjacent to $v$, and $v(0, i)$ ($0 \leq i < d - 1$) denotes the unique node in $S_{d-1}$ such that $\text{dim}(v(0), v(0, i)) = i$ (i.e., for $v = 1a_{d-2}\cdots a_0$, $v(0) = 0a_{d-2}\cdots a_0$, and $v(0, i) = 0a_{d-2}\cdots \overline{a_i}\cdots a_0$ where $\overline{a_i}$ is the complement of $a_i$).

We construct an embedding $f$ of $Q_n$ in $V$. We first define $f(v) = D(v)$ for each node $v$ of $S_b$, and then recursively extend $f$ to each $S_d$ ($b < d \leq n$) such that the $d$ neighbors of each node of $S_d$ are evenly partitioned into $b$ groups with each group sharing one wavelength. Algorithm 2.1 gives precise construction of $f$. 

6
Algorithm 2.1  

**Embedding**

**Input:** $Q_n$, $b$ and $V$.

**Output:** An embedding $f$ of $Q_n$ in $V$.

```
for each $v \in S_b$ do
    $f(v) \leftarrow D(v)$
endfor

for $d = b + 1$ to $n$ do
    for each $v \in S_d - S_{d-1}$ do
        $f(v) \leftarrow f(v(0, (d-1) \mod b)) + 2^{d-1}$
    endfor
endfor

end Algorithm.
```

**Lemma 2.1:** The function $f$ obtained from Algorithm 2.1 constitutes an embedding of $Q_n$ in the network $V$ such that for any node $v$ of $Q_n$ and any two neighboring nodes $v_s$ and $v_t$ of $v$, $\dim(v, v_s) \equiv \dim(v, v_t) \pmod{b}$ if and only if $f(v_s) \equiv f(v_t) \pmod{k}$. Consequently, for any node of $Q_n$, its $n$ neighboring nodes are partitioned into $b$ groups consisting of $\alpha$ groups with size $\left\lfloor \frac{n}{b} \right\rfloor$ and $\beta$ groups with size $\left\lceil \frac{n}{b} \right\rceil$ such that all the receivers in each group share a wavelength.

**Proof:** The lemma clearly holds for $n = b$. We proceed by induction on $n > b$. Note that $f$ on $S_d$ is one-to-one for each $d$ ($b < d \leq n$) since $f$ on $S_b$ is one-to-one. Thus we need only prove the following statement.

(*) For any node $v$ of $Q_n$ and any two neighboring nodes $v_s$ and $v_t$ of $v$,

$$\dim(v, v_s) \equiv \dim(v, v_t) \pmod{b} \text{ if and only if } f(v_s) \equiv f(v_t) \pmod{k}.$$  

Let $v \in Q_n$ and $v_s, v_t \in A(Q_n : v)$ ($v_s \neq v_t$). First let $n = b + 1$. Suppose $\dim(v, v_s) \equiv \dim(v, v_t) \pmod{b}$. Then since $|\dim(v, v_s) - \dim(v, v_t)| = b$, we may assume $\dim(v, v_s) = 0$ and $\dim(v, v_t) = b$. If $v \in S_b$ then $f(v_t) = f(v_s) + 2^b$ (since
\( v_s = v_t(0,0) \), while if \( v \not\in S_b \) then \( f(v_s) = f(v_t) + 2^b \) (since \( v_t = v_s(0,0) \)). Hence \( f(v_s) \equiv f(v_t) \mod k \). Conversely, suppose \( f(v_s) \equiv f(v_t) \mod k \). Then only one of \( v_t, v_s \) must be a node of \( S_b \), say \( v_s \) is a node of \( S_b \). Since \( f(v_t) = f(v_1(0,0)) + 2^b \), it follows that \( f(v_1(0,0)) = f(v_s) \) and thus \( v_1(0,0) = v_s \). Now, if \( v \in S_b \) then \( \dim(v, v_t) = b \) and \( \dim(v, v_s) = 0 \), while if \( v \not\in S_b \) then \( \dim(v, v_t) = 0 \) and \( \dim(v, v_s) = b \). Thus \( \dim(v, v_s) \equiv \dim(v, v_t) \mod b \). This establishes the statement (*) for \( n = b + 1 \).

Next, let \( n > b + 1 \) and assume the statement (*) for \( Q_{n-1} \). Write \( r = (n - 1) \mod b \).

Case 1. \( v \in S_{n-1} \)

We need only consider the case where only one of \( v_s, v_t \) is a node of \( S_{n-1} \). So, let \( v_s \in S_{n-1} \) and \( v_t \not\in S_{n-1} \). Note that \( \dim(v, v_t) \equiv \dim(v, v_t(0,r)) \mod b \) and \( f(v_t) \equiv f(v_t(0,r)) \mod k \). Now by applying the inductive hypothesis to the two neighboring nodes \( v_s \) and \( v_t(0,r) \) of \( v \) in \( S_{n-1} \), the statement (*) in this case follows.

Case 2. \( v \not\in S_{n-1}, v_s \not\in S_{n-1} \) and \( v_t \not\in S_{n-1} \).

Note that \( v_s(0,r) \) and \( v_t(0,r) \) are neighboring nodes of \( v(0,r) \) in \( S_{n-1} \), further \( \dim(v, v_s) = \dim(v(0,r), v_s(0,r)) \) and \( \dim(v, v_t) \equiv \dim(v(0,r), v_t(0,r)) \). The statement (*) in this case follows by applying the inductive hypothesis to the two neighboring nodes \( v_s(0,r) \) and \( v_t(0,r) \) of \( v(0,r) \) in \( S_{n-1} \).

Case 3. \( v \not\in S_{n-1} \) and only one of \( v_s, v_t \) is a node of \( S_{n-1} \).

Let \( v_s \in S_{n-1} \) and \( v_t \not\in S_{n-1} \). Since \( v_s = v(0) \), we see that \( \dim(v, v_t) = \dim(v_t(0), v_s) \) and \( \dim(v, v_s) \equiv \dim(v_t(0), v_t(0,r)) \mod b \). The statement (*) in this case follows by applying the inductive hypothesis to the two neighboring nodes \( v_t(0,r) \) and \( v_s \) of \( v_t(0) \) in \( S_{n-1} \).

This completes the proof of the statement (*) in all cases, and the proof is complete.
Under the embedding $f$ of $Q_n$ obtained from Algorithm 2.1, we can determine the wavelength assignment to the receivers in $Q_n$ in the following way. Let $v = a_{n-1}a_{n-2} \cdots a_0$ be a node of $Q_n$. Denote by $w(v)$ the wavelength assignment to the receiver of the node $v$, and write $q = \left\lfloor \frac{n}{b} \right\rfloor$. Then Algorithm 2.1 together with Lemma 2.1 yields that $w(v) = w_i$ with $i = D((a_{q(b-1)}a_{q(b-2)} \cdots a_{q}) \oplus (a_{q(b-1)}a_{q(b-2)} \cdots a_{q-1})) \oplus \cdots \oplus (a_{2b-1}a_{2b-2} \cdots a_{b}) \oplus (a_{b-1}a_{b-2} \cdots a_0))$, where $a_j = 0$ for $j > n - 1$ and $\oplus$ is the bit-by-bit exclusive-or operation. Figure 2 shows the embedding $f$ and the wavelength assignment to the receivers in $Q_n$ for $n = 5$ and $b = 3$.

In the rest of this paper, we will assume the embedding $f$ of $Q_n$ in the network; accordingly, the node set $\{v_i \mid 0 \leq i < N\}$ of $Q_n$ will also denote the node set $V$ of the network without any notational confusion.

3 Transmission Schedules

In this section, we construct an optimal transmission schedule for each $\delta$. First we will design an algorithm, called Initial Allocation ($Q_n$), for initial allocation schedules (satisfying the conditions $(i \cdot iv)$ preceding Algorithm 3.1) for a cycle of $\frac{nN}{k}$ consecutive time slots. Next, we will construct an optimal transmission schedule for each $\delta$ by modifying the initial allocation schedule obtained from Initial Allocation ($Q_n$).

3.1 Initial Allocation Schedules

In order to facilitate our discussion we need to introduce the following notation and a partition of the node set $V$, together with matrices $M$ and $Q$ which will be used as indicators for the construction of an initial allocation schedule. Let $P = \{V_i \mid 0 \leq i < \frac{N}{k}\}$ be a partition of the node set $V = \{v_i \mid 0 \leq i < N\}$ where $V_i = \{v_j \mid ik \leq j < (i + 1)k\}$, and write $W_q = \{v_i \in V \mid r_i \in R_q\} (0 \leq q < k)$. Note that
Figure 2: Embedding $f$ of $Q_n$ in the network for $n = 5$ and $b = 3$.

$v_i \in W_q$ if and only if $q \equiv i \pmod{k}$, and that $|V_i \cap W_q| = 1$ (i.e., $V_i$ contains exactly one receiver in $R_q$) for each $i$ and $q$ ($0 \leq i < \frac{N}{k}$, $0 \leq q < k$).

Our initial allocation schedule will be constructed to satisfy the following conditions:

1. For each time slot, $k$ nodes in some $V_i$ ($0 \leq i < \frac{N}{k}$) transmit simultaneously.

2. For each $v \in V$, $v$ transmits once to each of its $n$ neighboring nodes within a repeating cycle using $\alpha$ blocks of consecutive time slots of length $\lfloor \frac{n}{b} \rfloor$ and $\beta$ blocks of consecutive time slots of length $\lceil \frac{n}{b} \rceil$.

Thus the cycle length of initial allocation schedule is $\frac{nN}{k}$ (total transmission time divided by $k$). Note that for each $\delta$ a transmission schedule can be constructed from the initial allocation schedule by adding extra time slots for a large $\delta$ or by replacing redundant tunings with blanks for a small $\delta$.

A schedule (initial allocation schedule or transmission schedule) will be expressed as a table that represents a repeating cycle of time and wavelength allocations. Each entry $(s, l)$ of the table occurring in the row $s$ and the column $l$ is either $j$, $w_d$ or blank: during the time slot $l$, $(s, l) = j$ if $t_s$ transmits to $r_j$, $(s, l) = w_d$ if $t_s$ tunes to wavelength $w_d$, and $(s, l) = blank$ if $t_s$ is idle. Time slots between two adjacent transmission blocks for each $t_s$ can be used to tune to wavelength $w_d$.

As an example, consider a network with 32 nodes and 8 wavelengths and a 5-dimensional hypercube $Q_5$ as a virtual topology with the embedding $f$ (see Figure 2). A possible initial allocation schedule of cycle length $\frac{nN}{k} = 20$ is shown in Figure
3(a). Figure 3(b) shows a transmission schedule of cycle length 22 for \( \delta = 5 \), in which one or two extra time slots are added to the initial allocation schedule for some transmitters to satisfy the \( \delta \) tuning time while an idle time slot is assigned to some other transmitters. In fact, two different sizes of transmission blocks and their arrangement in the initial allocation schedule in Figure 3(a) cause for the extra time slots in Figure 3(b). Figure 6 illustrates another initial allocation schedule (for \( n = 5 \) and \( b = 3 \)) of cycle length 20 with rearranged \( b \) transmission blocks for each \( V_i \) \( (0 \leq i < \frac{N}{k}) \) which is also a feasible transmission schedule for \( \delta = 5 \). Thus the transmission schedule in Figure 3(b) obtained from the initial allocation schedule in Figure 3(a) is not optimal. We conclude that an arrangement of \( b \) transmission blocks for each \( V_i \) \( (0 \leq i < \frac{N}{k}) \) is critical to the cycle length of a schedule when \( \delta \) is in certain range.

In the following, we will construct an \( \frac{N}{k} \times b \) matrix \( M = (m_{ij}) \) which indicates an arrangement of the \( \alpha + \beta = b \) transmission blocks for each \( V_i \in P \) where \( m_{ij} = 0 \) or 1 such that \( m_{ij} = 1 \) if and only if \( m_{i-1} ((i+1) \mod b) = 1 \) \( (i > 0) \). If \( m_{ij} = 0 \) \( (m_{ij} = 1 \), respectively), it indicates that nodes in \( V_i \) transmit simultaneously using \( \left[ \frac{n}{b} \right] \left( \left[ \frac{n}{b} \right], \right. \) respectively) consecutive time slots. Therefore, each row of \( M \) must consist of \( \beta \) entries 1 and \( \alpha \) entries 0.

First we construct, for each \( l \) \( (0 \leq l < b) \), an \( \frac{N}{k} \times b \) matrix \( M(l) = (e_{ij}) \) \( (0 \leq i < \frac{N}{k}, 0 \leq j < b) \) with \( e_{ij} \in \{0, 1\} \) by: \( e_{ij} = 1 \) if and only if \( i + j \equiv \frac{N}{k} - 1 + l \) \( (\mod b) \). Note that for each \( l \), \( M(l) \) has exactly one entry 1 in each row and zeros elsewhere, and further, \( e_{ij} = 1 \) if and only if \( e_{i-1} ((i+1) \mod b) = 1 \) \( (i > 0) \). Also, note that for \( M(l_1) = (e_{ij}) \) and \( M(l_2) = (e'_{ij}) \) with \( l_1 \neq l_2 \) \( (0 \leq l_1, l_2 < b) \), \( e_{ij} \cdot e'_{ij} = 0 \) for every \( i, j \). We now define an \( \frac{N}{k} \times b \) matrix \( M = (m_{ij}) \) \( (0 \leq i < \frac{N}{k}, 0 \leq j < b) \) by:

**Case 1.** \( c \leq 1 \). Define \( M \) by \( M = \sum_{i=0}^{\beta-1} M(l) \).

**Case 2.** \( c > 1 \). Let \( d \) and \( e \) be, respectively, the greatest common divisor and the
### Figure 3 (a):
The image provides a schedule for $n = 5$, $b = 3$.

![Time slots]

### Figure 3 (b):
The image provides another schedule for $n = 5$, $b = 3$ with $\delta = 5$.

![Time slots]
least common multiple of \( c - 1 \) and \( b \). Write \( e = t(c - 1) \), and note that \( b = dt \). Let \( g_0 : \{0, 1, 2, \ldots, t - 1\} \to \{0, 1, 2, \ldots, b - 1\} \) be a function given by \( g_0(i) = i(c - 1) \mod b, 0 \leq i \leq t - 1 \). To show that \( g_0 \) is injective, suppose \( g_0(i) = g_0(i') \). Then \( t \) divides \( s(i - i') \) where \( c - 1 = ds \). Since \( t \) and \( s \) are relatively prime, it follows that \( t \) divides \( i - i' \), and so \( i = i' \).

Hence \( g_0 \) is injective. Next, define a function \( g \) from \( \{0, 1, 2, \ldots, b - 1\} \) to itself by: \( g(x) = (g_0(j) + i) \mod b \) for \( 0 \leq x < b \), where \( x = it + j \) \((0 \leq i < d, 0 \leq j < t)\). Note that \( g = g_0 \) on \( \{0, 1, 2, \ldots, t - 1\} \), and it follows that \( g \) is a permutation on \( \{0, 1, 2, \ldots, b - 1\} \). Now, define \( M \) by \( M = \sum_{l=0}^{\beta - 1} M(g(l)) \). For example, the permutation \( g \) for \( b = 9 \) with \( c = 7 \) is

\[
g = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 0 & 6 & 3 & 1 & 7 & 4 & 2 & 8 & 5 \end{pmatrix}
\]

Figure 4 (a)-(e) shows a sequence of \( \beta(= 5) \) matrices \( M(l) \) \((l = g(i), 0 \leq i \leq 4)\) for \( n = 13 \) and \( b = 9 \). The matrix \( M = (m_{ij}) \) constructed for each \( c \) has the property that each row of \( M \) has exactly \( \beta \) entries 1 and zeros elsewhere, further \( m_{ij} = 1 \) if and only if \( m_{i-1 \mod b, j+1} = 1(i > 0) \). In particular, if \( \alpha = 0 \), then each entry of \( M \) is 1. Also observe that \( m_{0 \ g(i)} = m_{\frac{x-1}{c} \ g(i)} = 1 \) \((0 \leq i < \beta - 1)\) if \( c = 0 \), \( m_{0 \ g(i)} = m_{\frac{x-1}{c} \ g(i)} = 1 \) \((0 \leq i < \beta - 1)\) if \( c = 1 \), and \( m_{0 \ g(i)} = m_{\frac{x-1}{c} \ g(i)} = 1 \) \((1 \leq i < \beta - 1)\) if \( c > 1 \). It
follows that $|\sum_{t>0} m_{it} - \sum_{t>0} m_{ij}| \leq 1$ for each $i, j \ (0 \leq i, j < b)$.

It follows by Lemma 2.1 that for each node $v_s \in V$, $A(Q_n : v_s)$ is partitioned into $b$ groups $U_{s0}, \cdots, U_{s(b-1)}$ where $U_{sl} = \{u \in A(Q_n : v_s) \mid \text{dim}(v_s, u) \mod b = l\}$ $(0 \leq l < b)$. Note that $|U_{sl}| = \left\lceil \frac{n}{b} \right\rceil$ for $0 \leq l < \alpha$, and $|U_{sl}| = \left\lfloor \frac{n}{b} \right\rfloor$ for $\alpha \leq l < b$, and that all the receivers in each group $U_{sl}$ $(0 \leq l < b)$ share a wavelength. Let $p_{sl}$ $(0 \leq s < N, \ 0 \leq l < b)$ denote the subscript of the wavelength shared by all the receivers in $U_{sl}$, so that the wavelength of each receiver in $U_{sl}$ is $w_{p_{st}}$. Since $U_{sl} = A(G : v_s) \cap W_{p_{st}}$ $(0 \leq l < b)$, we conclude that $|A(G : v_s) \cap W_{p_{st}}| = \left\lfloor \frac{n}{b} \right\rfloor$ for $0 \leq l < \alpha$, and $|A(G : v_s) \cap W_{p_{st}}| = \left\lceil \frac{n}{b} \right\rceil$ for $\alpha \leq l < b$.

Next, we construct an $N \times b$ matrix $Q = (q_{st}) \ (0 \leq s < N, \ 0 \leq l < b)$ which indicates transmissions of each $v_s \in V$ $(0 \leq i < \frac{N}{k})$ to the nodes in $A(Q_n : v_s)$ where $q_{st}$ indicates transmissions of $v_s$ to the nodes in $A(Q_n : v_s) \cap W_{q_{st}}$ using $\left\lceil \frac{n}{b} \right\rceil$ or $\left\lfloor \frac{n}{b} \right\rfloor$ consecutive time slots depending on whether $m_{ij}$ is 0 or 1. We use row $i$ of $M$ and $\{p_{sl} \mid ik \leq s < (i+1)k, \ 0 \leq l < b\}$ to construct $k$ rows (row $ik$ to row $(i+1)k-1$) of $Q$ as follows. Let $m_{i(0)}$, $m_{i(1)}$, $\cdots$, $m_{i(\alpha)}$, $m_{i(\alpha+1)}$, $\cdots$, $m_{i(b-1)}$ $(l(0) < l(1) < \cdots < l(\alpha-1); \ l(\alpha) < l(\alpha+1) < \cdots < l(b-1))$ be a rearrangement of row $i$ of $M$ such that $m_{i(j)} = 0$ for $0 \leq j < \alpha$, and $m_{i(j)} = 1$ for $\alpha \leq j < b$. Define entries of row $s$ $(ik \leq s < (i+1)k)$ of $Q$ by $q_{st} = p_{sj} \ (0 \leq j < b)$. It follows that if $m_{it} = 0$, then $|A(Q_n : v_s) \cap W_{q_{st}}| = \left\lceil \frac{n}{b} \right\rceil$ for each $v_s \in V$, while if $m_{it} = 1$, then $|A(Q_n : v_s) \cap W_{q_{st}}| = \left\lfloor \frac{n}{b} \right\rfloor$ for each $v_s \in V$. Since $0 \leq i < \frac{N}{k}$, the integers $\{q_{st} \mid 0 \leq s < N, \ 0 \leq l < b\}$ constitute the $N \times b$ matrix $Q = (q_{st}) \ (0 \leq s < N, \ 0 \leq l < b)$. Note that if $\alpha = 0$ then $q_{st} = p_{sl}$ for each $s, l \ (0 \leq s < N, \ 0 \leq l < b)$. Figure 5 illustrates the matrices $M$ and $Q$ for $n = 5$ and $b = 3$. We summarize the main results of the construction of $Q = (q_{st})$ and $M = (m_{ij})$ in the following lemma.

**Lemma 3.1:** (1) Each row of the matrix $M = (m_{ij})$ has exactly $\beta$ entries 1 and zeros elsewhere.
(2) For $i > 0$, $m_{ij} = 1$ if and only if $m_{i-1 \mod b} = 1$.

(3) If $m_{il} = 0$, then $|A(Q_n : v_s) \cap W_{q_{sl}}| = \left\lfloor \frac{n}{b} \right\rfloor$ for each $v_s \in V_i$.

(4) If $m_{il} = 1$, then $|A(Q_n : v_s) \cap W_{q_{sl}}| = \left\lceil \frac{n}{b} \right\rceil$ for each $v_s \in V_i$.

(5) $|\Sigma_{l>0} m_{il} - \Sigma_{l>0} m_{lj}| \leq 1$ for each $i, j \ (0 \leq i, j < b)$.

Note by Lemma 3.1 that the matrices $M$ and $Q$ can be used as indicators for an initial allocation schedule: $m_{il} = 0$ ($m_{il} = 1$, respectively) corresponds to transmissions of each $v_s \in V_i$ to the nodes in $A(Q_n : v_s) \cap W_{q_{st}}$ using $\left\lfloor \frac{n}{b} \right\rfloor$ ($\left\lceil \frac{n}{b} \right\rceil$, respectively) consecutive time slots.

Next, let $\{B_l \mid 0 \leq l < b\}$ be a partition of $\frac{nN}{k}$ consecutive time slots such that each $B_l$ is also consecutive time slots and the last time slot in $B_l$ is followed by the first time slot in $B_{l+1 \mod b}$, where each $|B_l|$ is given by $|B_l| = \sum_{i=0}^{b-1} (m_{il} \left\lfloor \frac{n}{b} \right\rfloor + (1 - m_{il}) \left\lceil \frac{n}{b} \right\rceil)$. Thus, each $B_l$ can be partitioned into $\frac{N}{k}$ groups, denoted by $\{B_{il} \mid 0 \leq i < \frac{N}{k}\}$, such that each $B_{il}$ is a consecutive time slots with $|B_{il}| = m_{il} \left\lfloor \frac{n}{b} \right\rfloor + (1 - m_{il}) \left\lceil \frac{n}{b} \right\rceil$ and, for each $i \ (0 \leq i < \frac{N}{k} - 1)$, the last time slot in $B_{il}$ is followed by the first time slot in $B_{i+1 \mod b}$.

Now, we construct an initial allocation schedule for a cycle of $\frac{nN}{k}$ consecutive time slots which satisfies the following conditions $(i-iv)$.

(i) For each $v_s \in V_i \ (0 \leq i < \frac{N}{k})$ and each block $B_l$, $v_s$ transmits to only the nodes in $A(G : v_s) \cap W_{q_{st}}$ using time slots $B_{il}$; $t_s$ tunes to wavelength $w_q$ (where $q = q_s ((l+1 \mod b))$) using all the time slots between the block of transmissions for $v_s$ in $B_l$ and the block of transmission for $v_s$ in $B_{(l+1 \mod b)}$. (We denote by $T_{st}$ the tuning allocations for $v_s$ between the block of transmissions for $v_s$ in $B_l$ and the block of transmission for $v_s$ in $B_{(l+1 \mod b)}$.)

(ii) For each time slot in $B_l$, exactly $k$ nodes which belong to some member of $P$ transmit simultaneously.

(iii) For each $l \ (0 \leq l < b)$, $|T_{il}| = |T_{jl}|$ for $i, j \ (0 \leq i, j < N)$.
(iv) For each $i$ and $j$ ($0 \leq i, j < b$), $||T_{0i}| - |T_{0j}|| \leq 1$.

The following algorithm which is based on Lemma 3.1 and the partition $\{B_l | 0 \leq l < b\}$ of $\frac{nN}{k}$ consecutive time slots gives an initial allocation schedule of each node $v_s$ in $V$ for each time block $B_l$. [Each tuning allocation $T_{sl}$ ($0 \leq s < N$, $0 \leq l < b$) in the initial allocation schedule will be modified to satisfy the $\delta$ tuning time requirement when we construct an optimal transmission schedule from the initial allocation schedule.]
Algorithm 3.1  Initial Allocation ($Q_n$)

**Input:** A hypercube $Q_n$ and $k$ ($N = 2^n$, $k = 2^b$).
**Output:** An initial allocation schedule $S_0$ of each node in $V$ for a cycle of $\frac{nN}{k}$ time slots.

for $l = 0$ to $b-1$ do
  /* schedule of each node in $V$ for time block $B_l$ */
  $\epsilon \leftarrow 0$;
  for $i = 0$ to $\frac{N}{k} - 1$ do
    /* schedule of each node in $V_i$ for time block $B_i$ */
    for $j = 0$ to $k - 1$ do
      $s \leftarrow ik + j$;  $q \leftarrow q_s$;
      $t_s$ tunes to wavelength $w_q$ using the first $\epsilon$ time slots in $B_i$;
      if ($m_{ij} = 0$) then /* $| A(Q_n : v_s) \cap W_q | = \left\lceil \frac{n}{b} \right\rceil$ */
        $v_s$ transmits to all nodes in $A(Q_n : v_s) \cap W_q$ using the next $\left\lceil \frac{n}{b} \right\rceil$ consecutive time slots in $B_i$
      else /* $| A(Q_n : v_s) \cap W_q | = \left\lfloor \frac{n}{b} \right\rfloor$ */
        $v_s$ transmits to all nodes in $A(Q_n : v_s) \cap W_q$ using the next $\left\lfloor \frac{n}{b} \right\rfloor$
        consecutive time slots in $B_i$
      endif
      $q \leftarrow q_{s((i+1) \mod b)}$;
      $t_s$ tunes to wavelength $w_q$ using the next remaining
      time slots in $B_i$
    endfor
    if ($m_{ij} = 0$) then $\epsilon \leftarrow \epsilon + \left\lceil \frac{n}{b} \right\rceil$ else $\epsilon \leftarrow \epsilon + \left\lfloor \frac{n}{b} \right\rfloor$ endif
  endfor
endfor
return ($S_0 \leftarrow$ schedule of each $v_s \in V$ for each $B_l$ (0 $\leq l < b$))
end Algorithm.

Figure 6 shows an initial allocation schedule obtained from Algorithm 3.1 for $n = 5$ and $b = 3$ (see the corresponding matrices $M$ and $Q$ in Figure 5).
Matrices $M$ and $Q$ for $n = 5$, $b = 3$.

(a) Initial allocation schedule from Algorithm 3.1 for $n = 5$ and $b = 3$. Each row $s$ ($0 \leq s \leq 31$) is the transmission schedule for $v_s \in V$: integral entry $l$ indicates that $v_s$ transmits to $v_l$ ($v_l \in A(Q_n : v_s)$); $w_i$ indicates that $t_s$ tunes to wavelength $w_i$.

(b) Block diagram of (a), $V_i = \{v_j \mid 8i \leq j < 8(i + 1)\}$ ($0 \leq i < 3$).
Lemma 3.2: The initial allocation schedule $S_0$ for a cycle of $\frac{nN}{k}$ time slots obtained from Algorithm 3.1 satisfies the conditions (i–iv).

Proof: Since each row $s$ of $Q$ is a permutation of $p_s, p_{s+1}, \ldots, p_{s+b-1}$, we see that the schedule $S_0$ transmits to all neighboring nodes of $v_s$ for each $v_s \in V$. The schedule $S_0$ clearly satisfies the condition (i). Since the first $\epsilon$ time slots in $B_i$ are used for tuning all nodes in $V_i$, it follows that for each time slot in $B_i$, exactly $k$ nodes which belong to some member of $P$ transmit simultaneously. The schedule $S_0$ thus satisfies the condition (ii). Next, Lemma 3.1 (2) yields that $|B_{i\ell}| = |B_{i-1 \mod b}(i+1)|$ for $i > 0$ (i.e., for any node $v \in V_i$, the number of time slots for transmitting $v$ in $B_i$ is equal to the number of time slots for transmitting any node of $V_i$ in $B_{i \mod b}$), which implies that $|T_{i\ell}| = |T_{j\ell}|$ for each $i$ and $j$ ($0 \leq i, j < N$). Hence the schedule $S_0$ satisfies the condition (iii). Finally, it follows from Lemma 3.1 (5) that $|T_{i0}| - |T_{i0}| \leq 1$ for each $i$ and $j$ ($0 \leq i, j < b$), thus the schedule $S_0$ satisfies the condition (iv).

This completes the proof of Lemma 3.2.

Remark: It follows from Lemma 3.2 that if $\delta \leq \min\{|T_{i0}| \mid 0 \leq l < b\}$ then $n + b\delta \leq \frac{nN}{k}$, while if $\delta > \min\{|T_{i0}| \mid 0 \leq l < b\}$ then $n + b\delta > \frac{nN}{k}$.

3.2 Optimal Transmission Schedules

First, we evaluate a lower bound for the cycle length of any optimal transmission schedule.

Lemma 3.3: The cycle length of any optimal transmission schedule is at least $\max\{\frac{nN}{k}, n + b\delta\}$.

Proof: Let $L$ denote the cycle length of any optimal transmission schedule. Since each node in $G$ has $n$ neighboring nodes, total $nN$ transmissions need to be done in a cycle. This implies that $L \geq \frac{nN}{k}$ since at most $k$ transmissions can be done simul-
taneously. On the other hand, observe that any node $v$ in $G$ has $n$ neighboring nodes whose receivers are tuned to $b$ different wavelengths. Therefore, we need tuning time at least $b\delta$ for the transmitter assigned to $v$ to tune to $b$ different wavelengths. Since there have to be $n$ transmissions from $v$, it implies that $L \geq n + b\delta$. We conclude that $L \geq \max\{\frac{nN}{k}, n + b\delta\}$. 

Now, we construct an optimal transmission schedule $\mathcal{S}$ by modifying the initial allocation schedule $\mathcal{S}_0$ obtained from Algorithm 3.1: For the construction of $\mathcal{S}$, the tuning allocations $T_{sl}$ ($0 \leq s < N$, $0 \leq l < b$) of the schedule $\mathcal{S}_0$ will be modified in order to satisfy the $\delta$ tuning time requirement by using additional time slots for a large $\delta$, or by replacing redundant tunings with blanks for a small $\delta$. Write $A = \min\{|T_{il}| \mid 0 \leq l < b\}$. (Note by Lemma 3.2 that $|T_{il}| = |T_{il}|$ for each $i$, $l$ ($0 \leq i < N$, $0 \leq l < b$).)

**Case 1.** $\delta \leq A$.

Let $\mathcal{S}$ be a schedule modified from the initial allocation schedule $\mathcal{S}_0$ by replacing the first $|T_{il}| - \delta$ allocations of $T_{sl}$ with blanks for each $s$, $l$ ($0 \leq s < N$, $0 \leq l < b$).

Note that the new schedule $\mathcal{S}$ constitutes a feasible transmission schedule, also that $\mathcal{S}$ has cycle length $\frac{nN}{k}$ since $\mathcal{S}$ and $\mathcal{S}_0$ are of the same cycle length. By Lemma 3.3, we conclude that $\mathcal{S}$ is an optimal transmission schedule.

**Case 2.** $\delta > A$.

Let $\mathcal{S}$ be a schedule modified from the initial allocation schedule $\mathcal{S}_0$ by the following:

For each $l$ ($0 \leq l < b$), insert $\delta - |T_{il}|$ time slots $u_{il}$ between $B_l$ and $B_{(l+1) \mod b}$ and then assign to each time slot in $u_{il}$ the contiguous tuning of $T_{sl}$ for each $s$ ($0 \leq s < N$).
Note that $T_{st}$ is extended to $\delta$ time slots for each $s$ and $l$ ($0 \leq s < N$, $0 \leq l < b$). Thus the new schedule $S$ has exactly $\delta$ tuning time between any two blocks of transmissions for each $v_s \in V$, which implies that the cycle length of $S$ is $n + b\delta$. By Lemma 3.3, we conclude that $S$ is an optimal transmission schedule.

Figure 7 shows the optimal transmission schedules $S$ for $n = 5$, $b = 3$ (a) with $\delta = 2$ and (b) with $\delta = 7$. (Each $B'_l$, $0 \leq l < b$, is the extended block of time slots from the block $B_l$ of the initial allocation schedule $S_0$, where $|B'_l| = |B_0| + \delta$.)

The above construction of $S$ gives the following lemma.

**Lemma 3.4:** The cycle length $L$ of any optimal transmission schedule is given by:

$$L = \frac{nN}{k} \text{ for } \delta \leq A \text{ and } L = n + b\delta \text{ for } \delta > A,$$

where $A = \min\{|T_0| \mid 0 \leq l < b\}$.

From Lemma 3.4 and the remark following the proof of Lemma 3.2, we establish the following theorem on optimal transmission schedules for hypercube interconnection.

**Theorem 3.1:** Consider the Optimal Transmission Schedule Problem assuming that (i) there are $k = 2^b$ wavelengths in the network, and (ii) $G$ is an $n$-dimensional hypercube with $N$ nodes. Then, the cycle length $L$ of any optimal transmission schedule is

$$L = \max\{\frac{nN}{k}, n + b\delta\}.$$

**Remarks:** (1) For given $k = 2^b$ wavelengths and tuning time $\delta$, each transmission schedule in hypercube interconnection is based on its underlying embedding of $n$-dimensional hypercube $Q_n$ in the (WDM optical passive star) network with $N = 2^n$ nodes. Thus an optimal transmission schedule is relative to its underlying embedding. As noted earlier, our initial allocation schedules and optimal transmission schedules are all based on the embedding $f$ constructed in Section 2.

(2) Let $H$ denote the set of all embeddings of $Q_n$ in the network. An optimal transmission schedule on some $g \in H$ is globally optimal if its cycle length is equal
Figure 7 (a):
Optimal transmission schedule $\mathcal{S}$ for $n = 5$, $b = 3$ with $\delta = 2$.

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Figure 7 (b):
Optimal transmission schedule $\mathcal{S}$ for $n = 5$, $b = 3$ with $\delta = 7$.

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to \( \min\{L_h \mid h \in H\} \), where \( L_h \) denotes the cycle length of an optimal transmission schedule on \( h \). Reasoning as in the first half of the proof of Lemma 3.3, we have \( L_h \geq \frac{nN}{k} \) for all \( h \in H \). Thus any transmission schedule (on some \( h \in H \)) of cycle length \( \frac{nN}{k} \) is globally optimal. Consequently, the optimal transmission schedules of cycle length \( \frac{nN}{k} \) constructed for small \( \delta \)'s (i.e., \( \delta \leq \min\{T_{\alpha} \mid 0 \leq l < b\} \)) in this section are globally optimal.

References


