Propagation of Uncertainty in a Maritime Risk Assessment Model utilizing Bayesian Simulation and Expert Judgment Aggregation Techniques

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1. ORGANIZATION

• Briefly review Bayesian Simulation of San Francisco Bay Exposure Analysis.


• Propagate simulation uncertainty and expert judgment uncertainty in a proof of concept case study using WSF expert judgment and SF bay exposure simulation.
2. WHY ADDRESS UNCERTAINTY?

Example of WSF Risk Assessment Results
Example of SF Bay Exposure Assessment Results
One problem with the representations in figures above is the apparent finality of the results. The decision-maker is led to believe that the results are definitive and are in no way uncertain. The National Research Council performed a peer review of the Prince William Sound Risk Assessment (the father and grand father risk assessment of the Washington State Ferry Risk Assessment and San Francisco Bay Exposure Assessment, respectively) and concluded that the underlying methodology shows "promise" to serve as a systematic approach for making risk management decisions for marine systems, but uncertainty in results needs to be addressed.

"Risk management ... should answer whether evidence is sufficient to prove specific risks and benefits"
(A. Elmer, President, SeaRiver Maritime, Inc. in National Research Council, 2000).

"Since the truth is, we always have uncertainty, we say that speaking in probability curves is telling the truth".
(see, e.g., Kaplan, 1997, p. 412)
1. BAYESIAN SIMULATION IN A NUTSHELL

Bayesian simulation differs from classical simulation analysis in that probability distributions are used to represent the uncertainty about model parameters rather than point estimates and confidence intervals. Such treatment is applied to both random inputs to the model and the outputs from the model.

Bayesian San Francisco Bay Exposure Analysis:

1. Bayesian Poisson process models of traffic arrivals were created for all 5,277 arrivals processes (excluding the Ferries since these run on a tight schedule).

2. For each arrival process a Gamma Prior Distribution was postulated for the arrival rate $\lambda$ of the exponential distributed inter arrival time.

3. For each arrival process a Gamma Posterior distribution for the arrival rate $\lambda$ was determined by a conjugate Bayesian analysis utilizing the inter arrival data for that process.
4. In the simulation, for each arrival one first samples an arrival rate $\lambda$ from its Gamma distribution and next sample the inter arrival time from the exponential distribution utilizing the sampled arrival rate $\lambda$, etc.

HENCE, **BAYESIAN SIMULATION TECHNIQUES**

ADDRESS BOTH:

**THE RANDOMNESS WITHIN THE SYSTEM UNDER CONSIDERATION**

(I.E. ALEATORY UNCERTAINTY)

AND

**LACK OF KNOWLEDGE ABOUT THE SYSTEM**

(I.E. EPISTEMIC UNCERTAINTY)
5. As our output data is in the form of a count (i.e. number of interactions per grid cell or total number of interaction), this number of vessel interactions can be naturally modeled using a Poisson distribution with rate $\lambda$, with a conjugate prior gamma distribution with shape $\alpha$ and scale $\gamma$.

6. For $s$ replications the posterior distribution of the number of vessel interactions is again a Poisson distribution with rate $\lambda$, with a conjugate posterior gamma distribution with shape $\alpha + \sum_{j=1}^{s} n_j$ and scale $\gamma + s$, where $n_j$ is the number of counts observed in the $j$-th replication and $s$ is the total number of (independent) replications of the simulation.

7. Our predictive distribution for the number of interactions (per grid cell) is then a Poisson Gamma distribution.

$$Pr(X = x) = \left( \frac{\gamma + s}{\gamma + s + 1} \right)^{\alpha + \sum_{j=1}^{s} n_j} \frac{\Gamma(\alpha + \sum_{j=1}^{s} n_j + x)}{\Gamma(x)\Gamma(\alpha + \sum_{j=1}^{s} n_j)} \frac{1}{x(\gamma + s + 1)^x}, x = 0, 1, 2, ...$$

See, e.g., Bernardo and Smith (1994), *Bayesian Theory*, Wiley
3. SF BAY EXPOSURE ASSESSMENT RESULTS WITH UNCERTAINTY

After 1 day of simulation
3. SF BAY EXPOSURE ASSESSMENT RESULTS WITH UNCERTAINTY

After 50 years of Simulation

Posterior 5-th percentile

Posterior 50-th percentile

Posterior 95-th percentile

48.24% of Total Base Case, Interactions

48.29% of Total Base Case, Interactions

48.33% of Total Base Case, Interactions

% of Max Exposure in:

Base Case

Factor X Average Exposure in:

Base Case

100.00 - 100.00

20.31 % 5.35

16.01 % 4.22

13.39 % 3.53

11.48 % 3.02

9.97 % 2.63

8.72 % 2.30

7.65 % 2.01

6.71 % 1.77

5.87 % 1.55

5.12 % 1.35

4.43 % 1.17

3.80 % 1.00

3.21 % 0.85

2.67 % 0.70

2.16 % 0.57

1.68 % 0.44

1.22 % 0.32

0.80 % 0.21

0.59 % 0.10

0.00 % 0.00
4. AGGREGATE SF BAY EXPOSURE ASSESSMENT RESULTS WITH UNCERTAINTY

A

6.42E+05

6.41E+05

6.40E+05

6.39E+05

# Yearly Situations

Base Case

B

9.76E+05

9.75E+05

9.74E+05

9.73E+05

# Yearly Situations

Alternative 3

C

7.20E+06

7.19E+06

7.19E+06

7.18E+06

# Yearly Situations

Alternative 2

D

1.26E+07

1.26E+07

1.26E+07

1.26E+07

# Yearly Situations

Alternative 1
5. AGGREGATE SF BAY EXPOSURE ASSESSMENT COMPARISON RESULTS WITH UNCERTAINTY

Box plots A-D of previous slide in a single graph

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6. CONCLUSION OF BAYESIAN SIMULATION OF SF BAY MTS IN A NUTSHELL

The uncertainty in the random arrival patterns of non-ferry traffic does not seem to affect in this particular study the conclusions regarding the different expansion scenarios (Base Case, Alternative 3, Alternative 2 and Alternative 1) from an Exposure Perspective. (which could be explained by the fact that interactions are predominantly FERRY to FERRY interactions which run on a tight schedule).

For more detailed information see:
7. ACCIDENT PROBABILITY ASSESSMENT USING BAYESIAN PAIRED COMPARISON ELICITATION

• An important class of elicitation techniques consists of the psychological scaling models that use the concept of paired comparisons. Origins can be traced back to Thurstone's (1927) and Bradley (1953)).

• Another popular paired comparison elicitation technique is called the Analytical Hierarchy Process (AHP) developed by Saaty (1977, 1980). The AHP Process is primarily used for the construction of value functions $V(\overline{X})$ involving multiple contributing factors $\overline{X} = (X_1, X_2, \ldots, X_p)$ (see, e.g. Foreman and Selly (2002)).

• The popularity of the paired comparison method can perhaps be contributed to the observation that experts are more comfortable making comparisons rather than directly assessing a quantity of interest.
• To the best of our knowledge, Pulkkinen (1993, 1994) was first to introduce a Bayesian paired comparison aggregation method for the elements of a multivariate random vector $\mathbf{\beta} = (\beta_1, \beta_2, \ldots, \beta_p)$ by multiple experts. Pulkkinen's (1993, 1994) exposition is mainly theoretical and limited to a discussion of mathematical properties.

• Similar to the AHP process, we are interested in the functional relationship between contributing factors $\mathbf{X} = (X_1, X_2, \ldots, X_p)$ and an accident probability $Pr(Accident|Incident, \mathbf{X})$ defined by

$$Pr(Accident|Incident, \mathbf{X}) = P_0 Exp(\mathbf{\beta}^T \mathbf{X}).$$

(1)

• $\mathbf{X} = (X_1, X_2, \ldots, X_p)$ describes a system state during which an incident (e.g. a mechanical failure) occurred.

• The accident probability model (1) resembles the well-known proportional hazards model originally proposed by Cox (1972) and builds on the assumption that accident risk behaves exponentially rather than linearly with changes in covariate values.
Our goal is to establish the uncertainty distribution of the accident probability \( Pr(Accident|Incident, X) \) in entirety rather than a point estimate.
### Table 1. Description of 10 contributing factors to $\Pr(\text{Accident} \mid \text{Incident}, X)$ in WSF Risk Assessment

<table>
<thead>
<tr>
<th>Designation</th>
<th>Description</th>
<th>Discretization</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_1$</td>
<td>FR_FC Ferry route-class combination</td>
<td>26</td>
</tr>
<tr>
<td>$X_2$</td>
<td>TT_1 1st interacting vessel type</td>
<td>13</td>
</tr>
<tr>
<td>$X_3$</td>
<td>TS_1 Scenario of 1st interaction</td>
<td>4</td>
</tr>
<tr>
<td>$X_4$</td>
<td>TP_1 Proximity of 1st interaction</td>
<td>$\text{Binary}$</td>
</tr>
<tr>
<td>$X_5$</td>
<td>TT_2 2nd interacting vessel type</td>
<td>5</td>
</tr>
<tr>
<td>$X_6$</td>
<td>TS_2 Scenario of 2nd interaction</td>
<td>4</td>
</tr>
<tr>
<td>$X_7$</td>
<td>TP_2 Proximity of 2nd interaction</td>
<td>$\text{Binary}$</td>
</tr>
<tr>
<td>$X_8$</td>
<td>VIS Visibility</td>
<td>$\text{Binary}$</td>
</tr>
<tr>
<td>$X_9$</td>
<td>WD Wind direction</td>
<td>$\text{Binary}$</td>
</tr>
<tr>
<td>$X_{10}$</td>
<td>WS Wind speed</td>
<td>$\text{Continuous}$</td>
</tr>
</tbody>
</table>

- $X \in [0, 1]^p$, $\beta \in \mathbb{R}^p$ and $P_0 \in (0, 1)$. The covariate $X_i$, $i = 1, \ldots, p$ are normalized so that $X_i = 1$ describes the "worst" case scenario and $X_i = 0$ describes the "best" case scenario.
Relative Scale of Concern

Vessel Class

- Naval Vessel
- Roll-On/Roll-Off
- Other
- Tanker
- Refrigerated Cargo
- Bulk Carrier
- Container
- Freight Ship
- Tug Boat/Barge
- Passenger
- High Speed Ferry
- Small Ferry
- Large Ferry

Constructing Covariate Scale for Interacting Vessels
### Question: 32

<table>
<thead>
<tr>
<th>Situation 1</th>
<th>Attribute</th>
<th>Situation 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Super</td>
<td>Ferry Class</td>
<td>-</td>
</tr>
<tr>
<td>SEA-BAI</td>
<td>Ferry Route</td>
<td>-</td>
</tr>
<tr>
<td>Naval Vessel</td>
<td>1st Interacting Vessel</td>
<td>-</td>
</tr>
<tr>
<td>Crossing the bow</td>
<td>Traffic Scenario 1st Vessel</td>
<td>-</td>
</tr>
<tr>
<td>1 to 5 miles</td>
<td>Traffic Proximity 1st Vessel</td>
<td>-</td>
</tr>
<tr>
<td>Deep Draft</td>
<td>2nd Interacting Vessel</td>
<td>-</td>
</tr>
<tr>
<td>Crossing the bow</td>
<td>Traffic Scenario 2nd Vessel</td>
<td>-</td>
</tr>
<tr>
<td>1 to 5 miles</td>
<td>Traffic Proximity 2nd Vessel</td>
<td>-</td>
</tr>
<tr>
<td>more than 0.5 mile</td>
<td>Visibility</td>
<td>less than 0.5 mile</td>
</tr>
<tr>
<td>Along Ferry</td>
<td>Wind Direction</td>
<td>-</td>
</tr>
<tr>
<td>40 knots</td>
<td>Wind Speed</td>
<td>-</td>
</tr>
</tbody>
</table>

| 9 8 7 6 5 4 3 2 1 2 3 4 5 6 7 8 9 |

Situation 1 is worse  \[ \Rightarrow \] Situation 2 is worse

---

An example question appearing in one of the questionnaires used in the WSF risk assessment

\[
P(X^1, X^2 | \beta) = \exp\left\{ \beta^T (X^1 - X^2) \right\} \in [0, \infty]. \tag{2}
\]

\[
\log\left\{ P(X^1, X^2 | \beta) \right\} = \beta^T (X^1 - X^2) \in (-\infty, \infty) \tag{3}
\]
8. THE LIKELIHOOD OF A SINGLE EXPERT'S RESPONSE

\[ Y_j = \text{Experts response to ratio} \quad \frac{Pr(Accident|Incident, X_j^1)}{Pr(Accident|Incident, X_j^2)}, \]

\[ Z_j = \log Y_j, j = 1, \ldots, n. \]

The response of the expert to such a question is uncertain and will assumed to be normal distributed such that

\[ (Z_j | \mu_j, r) \sim N(\mu_j, r), \quad r = 1/\sigma^2 \]

(4)

\[ \mu_j = q_j^T \beta, \quad q_j = (X_j^1 - X_j^2) \]

(5)

\[ f_{Z_j}(z_j) \propto \sqrt{r} \exp\left\{ -\frac{r}{2}(z_j - \mu_j)^2 \right\}. \]

(6)
• Expert answers *n paired comparison questions* defined by \( q_j = (X_j^1 - X_j^2) \), \( j = 1, \ldots, n \), Define \( Q \) to be the \( p \times n \) matrix and \( Z \) to be the vector with log responses of expert

\[
Q = [q_1, \ldots, q_n], \ Z = (z_1, \ldots, z_n).
\]  
(7)

• **Likelihood of an expert responding** \( Z \) **to questionnaire** \( Q \), may be derived from (6) as being proportional to

\[
\mathcal{L}(Z | \beta, r, Q) \propto r^n \exp \left\{ - \frac{r}{2} (c - 2 b^T \beta + \beta^T A \beta) \right\}.
\]  
(9)

where

\[
A = \sum_{j=1}^n q_j q_j^T; \ b = \sum_{j=1}^n q_j z_j; \ c = \sum_{j=1}^n z_j^2
\]  
(10)

If columns of \( Q \) span \( \mathbb{R}^p \) the matrix \( A \) can be shown to be symmetric, positive definite and henceforth invertible.
9. PRIOR DISTRIBUTION

• To allow for a conjugate Bayesian analysis a multivariate normal/gamma prior is proposed for the joint distribution of \((\boldsymbol{\beta}, r)\) similar to the one described in West and Harrison (1989).

\[
\Pi(r | \alpha, \nu) = \frac{\nu^{\alpha \frac{\alpha}{2}}}{\Gamma(\alpha \frac{\alpha}{2})} r^{\alpha \frac{\alpha}{2} - 1} \exp\left( - \frac{r}{2} \nu \right), \text{ i.e. } \text{Gamma}\left(\frac{\alpha}{2}, \frac{\nu}{2}\right). \tag{11}
\]

\[
\Pi(\boldsymbol{\beta} | r) \propto r^{\nu \frac{\nu}{2}} \exp\left\{ - \frac{r}{2} (\boldsymbol{\beta} - \boldsymbol{m})^T \Delta (\boldsymbol{\beta} - \boldsymbol{m}) \right\}, \text{ i.e. } \text{MVN}(\boldsymbol{m}, r\Delta). \tag{12}
\]

Hence, the joint prior distribution on \((\boldsymbol{\beta}, r)\) follows from (11) and (12) to be

\[
\Pi(\boldsymbol{\beta}, r) \propto r^{\alpha \frac{\alpha}{2} - 1} \exp\left( - \frac{r}{2} \nu \right) \times r^{\nu \frac{\nu}{2}} \exp\left\{ - \frac{r}{2} (\boldsymbol{\beta} - \boldsymbol{m})^T \Delta (\boldsymbol{\beta} - \boldsymbol{m}) \right\}. \tag{13}
\]
• The marginal distribution of $\beta$ may be derived from (14), yielding

$$
\Pi(\beta) \propto \left[ 1 + \frac{1}{\nu}(\beta - m)^T \Delta (\beta - m) \right]^{-\frac{\alpha + p}{2}}
$$

(14)

and is recognized as a $p$-dimensional multivariate $t$-distribution with $\alpha$ degrees of freedom, location vector $m$ and precision matrix $\frac{\alpha}{\nu} \Delta$.

• From (14) and (3) follows that the log-relative probability $Log\{P(X^1, X^2 | \beta)\}$ has a prior $t$-distribution with mean and precision

$$
\hat{m}^T (X^1 - X^2), \frac{\alpha}{\nu} (X^1 - X^2)^T \Delta (X^1 - X^2)
$$

(15)

9.1. Prior Parameter Specification

• A prior chi-squared distribution with $\alpha$ degrees of freedom (equivalent to a gamma distribution $\text{Gamma}\left(\frac{\alpha}{2}, \frac{\nu}{2}\right)$ with $\nu = 1$) and $E[r | \alpha, \nu=1] = \alpha$. 
• The prior parameter $\alpha$ will be set equal to the **reciprocal of the variance** of an **expert responding at random** and depends on the scale that is used in the paired comparison questions to collect the expert responses.

$$
\alpha = E[r|\alpha, \nu=1] = \left\{ \frac{2}{17} \sum_{k=2}^{9} \{\text{Log}(k)\}^2 \right\}^{-1} \approx 0.380341. \quad (16)
$$

• For distribution of $(\beta|r)$ we may select a **location vector** and the **unit precision matrix**

$$
m = (0, \ldots, 0)^T, \Delta = \begin{pmatrix} 1 & \emptyset \\ \emptyset & 1 \end{pmatrix}, \quad (17)
$$

as long as the prior distribution on the relative accident probabilities (2) are flat.

• The **pdf of the relative accident probability** in our previous question is a **log-\(t\) distribution** (see, e.g., McDonald and Butler (1987)) with prior parameters

$$
m^T(X^1 - X^2) = 0, \alpha = 0.380341, \nu = 1, \delta_{ii} = (X^1 - X^2)^T \Delta (X^1 - X^2) = 4.
$$
Prior on ($\beta$, $r$) and $P(X^1, X^2 | \beta)$ of previous question
• The prior median of $P(X^1, X^2 | \beta)$ equals 1 (indicating indifference in collision likelihood between system states $X^1$ and $X^2$).

• A 50% credibility interval of $P(X^1, X^2 | \beta)$ in the figure above equals [0.181, 5.515]. A 75% credibility interval of $P(X^1, X^2 | \beta)$ equals [2.012 $\cdot 10^{-5}$, 4.971 $\cdot 10^4$] (which is quite wide).

Table 2. Interaction Variables associated with the contributing factors in Table 1.

<table>
<thead>
<tr>
<th>Name</th>
<th>Description</th>
<th>Discretization</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_{11}$</td>
<td>FR_FC $\cdot$ TT_1</td>
<td>Interaction 13</td>
</tr>
<tr>
<td>$X_{12}$</td>
<td>FR_FC $\cdot$ TS_1</td>
<td>Interaction 13</td>
</tr>
<tr>
<td>$X_{13}$</td>
<td>FR_FC $\cdot$ VIS</td>
<td>Interaction 4</td>
</tr>
<tr>
<td>$X_{14}$</td>
<td>TT_1 $\cdot$ TS_1</td>
<td>Interaction Binary</td>
</tr>
<tr>
<td>$X_{15}$</td>
<td>TT_1 $\cdot$ VIS</td>
<td>Interaction 13</td>
</tr>
<tr>
<td>$X_{16}$</td>
<td>TS_1 $\cdot$ VIS</td>
<td>Interaction 4</td>
</tr>
</tbody>
</table>
10. POSTERIOR ANALYSIS

Applying Bayes theorem utilizing the likelihood (9), the prior distribution (13) and it follows that the posterior distribution \( \Pi(\beta, r \mid \mathcal{Z}, Q) \) is proportional to

\[
\Pi(\beta, r \mid \mathcal{Z}, Q) \propto r^{\alpha+n-1} \exp \left\{ -\frac{r}{2} \left( 1 + c + m^T \Delta m \right) \right\} \times \\
\propto r^\frac{p}{2} \exp \left\{ -\frac{r}{2} \left( -2 [b + \Delta m]^T \beta + \beta^T [A + \Delta] \beta \right) \right\}.
\]

Defining \( \Delta^u \) to be \( \Delta^u = A + \Delta \) and implicitly defining \( m^u \) satisfying

\[
\begin{bmatrix} b + \Delta m \end{bmatrix}^T \beta = \begin{bmatrix} \Delta^u m^u \end{bmatrix}^T \beta
\]

for all \( \beta \), it follows that

\[
b + \sum m = \Delta^u m^u \Leftrightarrow m^u = \left( \Delta^u \right)^{-1} \left( b + \Delta m \right).
\]
Utilizing (20) and $\Delta^u = A + \Delta$ we derive from (18) that
\[
\prod (\beta, r | Z, Q) \propto r^{\frac{\alpha+n}{2}-1} \exp \left\{ -\frac{r}{2} \left( 1 + c + m^T \Delta m - \left[m^u\right]^T \Delta^u m^u \right) \right\} \times \\
r^{\frac{p}{2}} \exp \left\{ -\frac{r}{2} \left[ \beta - m^u \right]^T \Delta^u \left[ \beta - m^u \right] \right\}.
\] (21)

From (21) it follows that $(\beta | Z, Q) \sim MVN(\mu^u, r\Delta^u)$ where
\[
\begin{align*}
\Delta^u &= \sum_{j=1}^n q_j q_j^T + \Delta \\
m^u &= \left( \Delta^u \right)^{-1} \left( \sum_{j=1}^n q_j z_j + \Delta m \right)
\end{align*}
\] (30)

and $(r|Z, Q) \sim Gamma(\alpha^u/2, \nu^u/2)$ with
\[
\begin{align*}
\alpha^u &= \alpha + n \\
\nu^u &= \nu + \sum_{j=1}^n z_j^2 + m^T \Delta m - \left[m^u\right]^T \Delta^u m^u
\end{align*}
\] (31)
11. EXAMPLE FROM WSF RISK ASSESSMENT

• **8 Experts** were selected amongst WSF captains and WSF first mates who had extensive experience with all 13 different ferry routes over an extended period of time (more than 5 years). **Combination** of the responses of these 8 experts follows naturally by **exploiting the conjugacy of the analysis** in Section 3, 4 and 5 through **sequential updating**.

<table>
<thead>
<tr>
<th>Expert Index</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Response</td>
<td>5</td>
<td>5</td>
<td>3</td>
<td>9</td>
<td>7</td>
<td>9</td>
<td>3</td>
<td>0.5</td>
</tr>
</tbody>
</table>

• **During the WSF risk assessment in 1998** expert responses were aggregated by taking **geometric means of their responses** and using them in a **classical log linear regression analysis** approach to assess relative accident probabilities given by (2). **Classical point estimates** for the parameters $\beta_j, j = 1, \ldots, 16$ will be compared to their **Bayesian counterparts** following our Bayesian aggregation method.
• Expert were instructed to assume that a navigation equipment failure had occurred on the Washington State Ferry and were next asked to assess how much more likely a collision is to occur in Situation 1 (good visibility in previous question) as compared to Situation 2 (bad visibility in previous question) taking into account the value of all the contributing factors. **Total of 60 Questions.** The questions were **randomized** in order and were **distributed evenly over the 10 contributing factors** in Table 1 (i.e. 6 questions per changing contributing factor).

**11.1. The elements $A$, $b$ and $c$ of the likelihood given by (10)**

$$
A = \begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix}
$$

(32)

where $A_{11}$ is a $10 \times 10$ diagonal matrix with diagonal elements

$$(4.56, 4.33, 2.89, 6, 1.5, 2.44, 6, 6, 6, 0.375)$$

(33)

and associated with the contributing factors $X_1, \ldots, X_{10}$. (The matrix $A_{11}$ in (32) is a diagonal matrix since the paired comparison scenarios $X^1$ and $X^2$ only
differed in one covariate (see, e.g., the previous question). The matrix $A_{22}$ in (32) is a symmetric $6 \times 6$ matrix with elements

\[
\begin{pmatrix}
3.45 & 0.33 & 0 & 1.44 & 0.76 & 0 \\
0.33 & 3.45 & 0.44 & 0.33 & 0 & 1 \\
0 & 0.44 & 4.11 & 0 & 1 & 2.39 \\
1.44 & 0.33 & 0 & 1.89 & 0.36 & 0.08 \\
0.76 & 0 & 1 & 0.36 & 3.02 & 2 \\
0 & 1 & 2.39 & 0.08 & 2 & 6.67
\end{pmatrix}
\]

and associated with the interaction effects $X_{11}, \ldots, X_{16}$. Finally, the matrix $A_{21} = A_{12}^T$ is a sparse $10 \times 6$ matrix

\[
\begin{pmatrix}
1 & 2.82 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2.26 & 0 & 2.12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1.13 & 0 & 0 & 0 & 0 & 0 & 3.06 & 0 & 0 & 0 \\
0 & 2.13 & 0.52 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1.02 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 1.56 & 0 & 0 & 0 & 0 & 5.33 & 0 & 0
\end{pmatrix}
\]

with only positive elements associated with the contributing factors $X_1, X_2, X_3$ and $X_8$ that are included in the interaction effects $X_{11}, \ldots, X_{16}$. 
Summary of Individual Expert Response for 8 WSF experts in terms of \( i \)-th element of the vector \( b \) (cf. (11)) for each of the contributing factors \( X_i, i = 1, \ldots, 10 \) in Table 1 and interaction effects \( X_i, i = 11, \ldots, 16 \) in Table 2.
Table 4. Values for $c$ (cf. (11)) for the 8 individual experts.

<table>
<thead>
<tr>
<th>Expert Index</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c$</td>
<td>149.07</td>
<td>95.28</td>
<td>55.74</td>
<td>147.93</td>
<td>185.71</td>
<td>177.30</td>
<td>147.12</td>
<td>44.94</td>
</tr>
</tbody>
</table>

11.2. Posterior Analysis

The resulting posterior parameters for the precision $r \sim Gamma\left(\frac{\alpha^u}{2}, \frac{\nu^u}{2}\right)$ are

$$\alpha^u = 480.38, \ \nu^u = 530.95$$ (36)

The posterior distribution of the parameter vector $\beta$ is a multivariate $t$ distribution with location vector $m^u$ and precision matrix $\frac{\alpha^u}{\nu^u} \Delta^u$, where $\alpha^u, \nu^u$ are given by (36),

$$\Delta^u = \Delta + 8A$$

and location vector $m^u$ is depicted in the following figure.
Comparison of Bayesian and Classical Point Estimates of the parameters $\beta_i$, $i = 1, \ldots, 16$. 
• It can thus be concluded that traffic proximity of the first and second interacting vessel \((X_4 \text{ and } X_7, \text{ respectively})\), traffic scenario of the second interacting vessel \(X_7\) and wind speed \(X_{10}\) are the largest contributing factors to accident risk. In addition, the manner in which the first interacting vessel approaches the ferry route - ferry class combination \((X_{12})\), i.e. crossing, passing or overtaking, and in what visibility conditions \((X_{16})\) are the largest interacting factors.

• A remarkable agreement should be noted between the Bayesian and classical point estimates provided in the figure above, except for a discrepancy associated with the contributing factor WS (Wind Speed).

• The next figure displays the posterior distribution of the relative probability \(P(X^1, X^2 | \beta)\) associated with our previous pair wise comparison question.
Posterior on $(\beta, r)$ and $P(X^1, X^2 | \beta)$ of Previous Question.
• The median point estimate of $P(X^1, X^2 | \beta)$ equals 4.94. Hence, Situation 2 in Figure 3 is approximately 5 times more likely to result in a collision than Situation 1 given that a navigation equipment failure occurred on the ferry.

• Compare the 50% posterior credibility interval of $P(X^1, X^2 | \beta)$ of [4.78, 5.13] to the 50% prior one of [0.18, 5.52]. In addition, the 99% posterior credibility interval of [4.33, 5.66] is indicated in the figure above which is remarkably narrow compared to the prior 75% credibility interval of [2.012 \cdot 10^{-5}, 4.971 \cdot 10^{4}]

• Utilizing posterior distributional results for the parameter vector $\beta$ credibility statements can be made for any arbitrary paired comparison. For example, setting Situation 1 in (2) to the best possible scenario ($X^1 = 0$) and Situation 2 to the worst possible scenario ($X^2 = 1$) a 99% credibility interval of $P(X^1, X^2 | \beta)$ equals [31142, 36749]. Therefore, collision risk in the worst possible scenario differs at least by 4 orders of magnitude to that of the best possible scenario while taking uncertainty of the expert judgments into account.
12. COMMENTS ON EXPERT JUDGMENT METHOD

• Bayesian aggregation method has been developed using responses from multiple experts to a paired comparison questionnaire to assess the distribution of relative accident probabilities. The classical analysis conducted during the WSF risk assessment only resulted in point estimates of relative accident probabilities.

For more detailed information see:

PREVIOUS EXPERT AGGREGATION METHOD UTILIZED SEQUENTIAL UPDATING WHICH ESSENTIALLY MEANS THAT GIVEN THE PARAMETER VECTOR $\beta$ AND THE PRECISION $\tau$ THE EXPERTS ARE STATISTICALLY INDEPENDENT.
THE LATTER ASSUMPTION CAN BE RELAXED! ESSENTIALLY BY UTILIZING A MULTIVARIATE BAYESIAN REGRESSION APPROACH TO THE PAIRED COMPARISON QUESTIONS INVOLVING A PRIOR MULTIVARIATE NORMAL ON \((\beta | \Sigma)\) WHERE \(\Sigma\) IS THE INVERSE VARIANCE COVARIANCE MATRIX AND A WISHART PRIOR OR \(\Sigma\).


**For more detailed information on our specific application see:**

• A *surprising* result is that the *incorporation of dependence* here results in a reduction of the *predictive variance* of the elements of the parameters vector \(\beta\)
90\% Creditibility Intervals of the elements of the parameter vector $\beta$ of $Pr(Accident|Incident, X) = P_0 Exp(\beta^T X)$
13. PUTTING IT ALL TOGETHER IN A PROOF OF CONCEPT CASE STUDY

![Graph showing the number of yearly accidents for different alternatives.](image-url)
Whereas an almost certain ranking in terms of the expected yearly number of situations, this is not true for the expected yearly number of accidents.

The box plots for the Base Case and Alternative 3 show that the range of their distributions do indeed overlap and the best we can say is that Alternative 3 stochastically dominates the Base Case in the sense that their cumulative distribution functions do not cross.
Reason for **Stochastic Dominance** and not **Deterministic Dominance** of the Alternative 3 and the Base Case is a reduction in the average accident probability per occurring situation.
Reason for a reduction in the average accident probability per occurring situation is a reduction in Alternative 3 of the number of second interacting vessels within a 1 mile distance. (Recall that Traffic Proximity of second interacting vessel was a dominant factor in the accident probability in a given interaction.)
TENTATIVE OBSERVATION:

The reduction in accident probability per interaction could be an indication of what might be achieved when looking at the SF Bay Ferry operation as a System of interconnected Ferry Routes and by designing a Comprehensive Ferry Schedule that aims to reduce the number of interactions and the number of vessels that are interacting in given situation.

For more detailed information on this proof of concept case study see:

QUESTIONS?
14. MARITIME RISK ASSESSMENT LINKS

Faculty Home Page of J. Rene van Dorp:
http://www.seas.gwu.edu/~dorpjr

and

Faculty Home Page of Jason R.W. Merrick:
http://www.people.vcu.edu/~jrmerric

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