The Doubly-Pareto Uniform Distribution with Applications in Uncertainty Analysis and Econometrics

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Abstract. In this paper we propose a novel four parameter continuous univariate distribution that can be motivated from at least two approached. The first one views the distribution as a generalization of the uniform one that allows for uncertainty specification at the vicinity of its bounds (gradually) represented via two Pareto tails. The second one is that of an asymmetric heavy-tailed, peaked distribution with an unbounded domain with the property that the location of the mode is not uniquely determined but rather is described by an uniform range. Properties of the distribution are described and a maximum likelihood estimation (MLE) procedure for the mode location and the Pareto tails parameters is presented. The procedure is illustrated by means of an i.i.d. sample of standardized log-differences of bi-monthly 30-year US certificate deposit interest rates for the period from 1964-2004. The sample is constructed utilizing the Auto-Regressive Conditional Heteroscedastic (ARCH) time series model devised by the Nobel Laureate R.F. Engle (1982).

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1. Introduction

One of the first records that specifically mentions the continuous uniform distribution seems to be the classical paper by the reverend Thomas Bayes (1763). While the uniform (rectangular) distributions are one of the corner stones of the methodology in mathematical statistics (e.g. integral transformations; Monte Carlo random number generation; non-parametric testing procedures), few citations have appeared in the relevant literature thus far dealing with applications of the uniform distribution per se. Hull and Swenson (1966) and more recently Wimmer and Witkovsky (2002) applied it to the problem of roundoff errors. Hlawka (1984) presents a comprehensive theory of the uniform distribution. Possibly the inherent simplicity of this distribution, which describes the equi-probability of the different possible values of uncertainty, hindered its direct applicability. To paraphrase A. Einstein: "The world is simple but not too simple".

With the proliferation of simulation and uncertainty analysis packages in the late 20-th century as a standard tool in an engineer's toolbox, undoubtedly, the most wide spread (indirect) application of the uniform distribution with support $[0, 1]$ seems to be to serve as a building block for sampling from continuous distributions utilizing the inverse cumulative distribution function theorem and a pseudo random number generator (see any of the widespread modern texts on simulation, e.g. Kelton et. al. (2002). Banks et al. (2001), Atliok and Melamed (2001)). Current uncertainty analysis packages like @Risk (by the Palisade Corporation) and Crystal Ball (by Decision Engineering) utilize this sampling method to analyze the distribution of an output parameter given a model description and input parameters with specified uncertainty distributions. The uniform distribution with arbitrary support $[a, b]$ is also a natural (initial) candidate for input uncertainty distribution specification, requiring just the assessment of a lower and upper bounds via an expert judgment (see, e.g., Ellison et al. (2000)).

From a statistical aspect, however, the elicitation of the minimal value $\hat{a}$ and the maximal value $\hat{b}$ of a bounded uncertain phenomenon from a substantive expert has been a procedure riddled with ambiguities and contradictions (See,e.g., Selvidge (1980), Davidson and Cooper (1980), Alpert and Raiffa (1982), Keefer and Verdini (1993)). The main objection is that the values to be assessed are quite likely to fall outside the range of the expert's experience in spite of his/her familiarity with the activity under consideration. To overcome this possible deficiency, Kotz and Van Dorp (2004) suggested to specify the p^{th} lower quantile (a_p) and the r^{th} upper quantile (b_r) of the uniform distribution $(0 < p < 1, 0 < r < 1)$ and then solving for its lower and upper bounds a and b via the relationships:

$$a = \frac{ra_p - pb_r}{r - p},\ b = \frac{(1-p)b_r - (1-r)a_p}{(r-p)}. \tag{1.1}$$

While (1.1) continues to adhere to a bounded support, the Doubly-Pareto Uniform (DPU) distribution, to be discussed herein, offers an unbounded alternative to (1.1) by modeling Pareto tails to the left of a_p and to the right of b_r, maintaining uniformity in between. Figures 1A and 1B in Section 2 display a DPU distribution with $a_{0.025} = 0$ and $b_{0.95} = 1$. Another possible direct application of the DPU distribution for uncertainty analysis arises by observing that its parameters may be assessed by specifying a range $[\alpha, \beta]$ for its modal value (rather than requiring a substantive expert to provide a fixed point estimate that he/she may be less comfortable with) and a lower quantile $a_p < \alpha$ and upper quantile $b_r > \beta$. Figures 1C and 1D display a DPU distribution with $\alpha = 0$ and $\beta = 1$, $a_{0.05} = -\frac{1}{4}$ and $b_{0.90} = 1\frac{1}{4}$. A third possible application arises from the reinforced confirmation that distributions of financial returns are strongly leptokurtic (see, e.g. Levy and Duchin

(2004), Kotz and van Dorp (2004), McFall Lamm (2003), Popova et al. (2003), Kotz et al. (2001)). This fact seems to have reinvigorated the search for continuous distributions of this type (see, e.g., Bardou et al. (2002), Huang and Solomon (2001), Matacz (2000), Solomon and Levy (2000) and Bouchaud et al. (1998), amongst others). The famous Swiss-Italian economist and sociologist Vilifredo Pareto over 100 years ago proposed Pareto distributions

$$f(x|\alpha,\beta,n) = \frac{n(\beta-\alpha)^n}{(x-\alpha)^{n+1}},$$

where $x > \beta > \alpha$, $n > 0$, with an eye towards modeling income distributions (see, e.g., Kleiber and Kotz (2003) for a recent source). These distributions have support $[0,\infty)$ and exhibit a power-law (or Paretian) behavior in its right tail and are known to be of the "heavy-tailed" type. The DPU distribution on the other hand has support $[-\infty,\infty]$, but due to its left and right Pareto tails is also of a heavy-tailed (or leptokurtic) kind.

The remainder of this paper is organized as follows: in Section 2 we present the pdf and cdf of the DPU family of distributions and some of its properties. Two elicitation procedures for the parameters of a DPU distribution from the two perspectives above are presented in Section 3. These procedures allow for estimation of DPU parameters in uncertainty analysis applications by means of expert judgment in the absence of data. For applications where data is available we derive a maximum likelihood procedure for estimating DPU parameters in Section 4. In Section 5 we apply the maximum likelihood (ML) procedure to the US certificate deposit interest rates data covering 1964-2004.

2. DPU distributions and properties

Let X be a random variable with the pdf

$$f(x|\alpha,\beta,m,n) = \frac{mn}{m+mn+n} \times \begin{cases} \frac{(\beta-\alpha)^m}{(\beta-x)^{m+1}}, & x < \alpha, \\ \frac{1}{\beta-\alpha}, & \alpha \le x \le \beta \ , \\ \frac{n(\beta-\alpha)^n}{(x-\alpha)^{n+1}}, & x > \beta, \end{cases} \tag{2.1}$$

where the parameters $\alpha < \beta$, $m > 0$, $n > 0$. Note that (2.1) can be rewritten as $f(x|\alpha,\beta,m,n) = \sum_{i=1}^{3} \pi_i f_{X_i}(x)$, where $f_{X_1}(x) = \frac{m(\beta-\alpha)^m}{(\beta-x)^{m+1}}$, for $x < \alpha < \beta$, $f_{X_2}(x) = \frac{1}{\beta-\alpha}$, for $\alpha \le x \le \beta$, $f_{X_3}(x) = \frac{n(\beta-\alpha)^n}{(x-\alpha)^{n+1}}$, for $x > \beta > \alpha$ and the weights $\pi_i, i = 1,\ldots,3,$ are given by

$$\begin{cases} \pi_1 = \frac{n}{m+mn+n}, \\ \pi_2 = \frac{mn}{m+mn+n}, \\ \pi_3 = \frac{m}{m+mn+n}. \end{cases} \tag{2.2}$$

Observe that $\pi_i > 0$, $i = 1,2,3$, f_{X_i}, $i = 1,3$ are Pareto distributions and f_{X_2} is a uniform distribution on $[\alpha,\beta]$. Hence the pdf $f(x|\alpha,\beta,m,n)$ follows the construction method of the modified mixture technique applied for example for the generalization of the trapezoidal distribution in Kotz and Van Dorp (2004). The pdf (2.1) will be called the Doubly-Pareto Uniform (DPU) distribution corresponding to its two Pareto tails and the central uniform stage. The parameters m and n will be referred to as the tail shape parameters of the DPU distribution and the parameters α and β the mode location (or centrality) parameters.

From (2.1) the cdf follows, using straightforward calculations, to be

$$F(x|\alpha,\beta,m,n)=\begin{cases}\frac{n}{m+mn+n}\left(\frac{\beta-\alpha}{\beta-x}\right)^{m}, & x<\alpha,\\ \frac{mn(x-\alpha)+n(\beta-\alpha)}{(m+mn+n)(\beta-\alpha)}, & \alpha\leq x\leq\beta\ ,\\ 1-\frac{m}{m+mn+n}\left(\frac{\beta-\alpha}{x-\alpha}\right)^{n}, & x>\beta.\end{cases} \tag{2.3}$$

Substituting $m=n>0$ in (2.1) yields the special case

$$f(x|\alpha,\beta,n)=\frac{n}{n+2}\times\begin{cases}\frac{(\beta-\alpha)^n}{(\beta-x)^{n+1}}, & x<\alpha,\\ \frac{1}{\beta-\alpha}, & \alpha\leq x\leq\beta\\ \frac{(\beta-\alpha)^n}{(x-\alpha)^{n+1}}, & x>\beta.\end{cases} \tag{2.4}$$

The derivation of the pdf (2.4) was, to the best of our knowledge, first described by DeGroot (1970) by defining a uniform distribution on $[a,b]$ with a bivariate bilateral Pareto distribution

$$g(a,b|\alpha,\beta,n)\ =\begin{cases}\frac{n(n+1)(\beta-\alpha)^n}{(b-a)^{n+2}} & a<\alpha,\ b>\beta\\ 0 & \text{elsewhere},\end{cases}$$

where $n>0$ and $\alpha<\beta$, describing uncertainty about the bounds $[a,b]$. Next the pdf (4) follows from

$$f(x|\alpha,\beta,n)=\int_{a=-\infty}^{\alpha}\int_{b=\beta}^{\infty}\frac{g(a,b|\alpha,\beta,n)}{b-a}\ 1_{[a,b]}(x)dadb,$$

where $1_{[a,b]}(x)$ is the indicator function that assumes the value one for $x\in[a,b]$ and zero otherwise. So far we have not been able to derive a similar bivariate bilateral pareto set-up for the more general pdf (2.1). While the pdf (2.4) is limited to symmetric pdfs with the symmetry axis $(\alpha+\beta)/2$, the pdf (2.1) possesses enhanced flexibility by allowing for asymmetry through seperate uncertainty specification of the left and right tails by means of the two parameters m and n.

Figure 1A (Figure 1B) plots an example DPU pdf (2.1) (cdf (2.3)), with parameters $m=37$, $n=18.5$, $\alpha=0$, $\beta=1$. Note that the central stage in Figure 1A is symmetric around $\frac{1}{2}$ and observe a linear behavior in $[0,1]$ in both Figures 1A and 1B. The distributions in Figures 1A and 1B are reminiscent of a uniform $[0,1]$ distribution with two very short tails describing remaining uncertainty beyond the bounds 0 and 1. Figure 1C (Figure 1D) plots an example DPU pdf (2.1) (cdf (2.3)), with parameters $m=4.645$, $n=3.203$, $\alpha=0$, $\beta=1$ exhibiting substantially longer tails than the distributions in Figures 1A and 1B. The distributions in Figure 1C and 1D resemble a unimodal assymetric distribution with unbounded

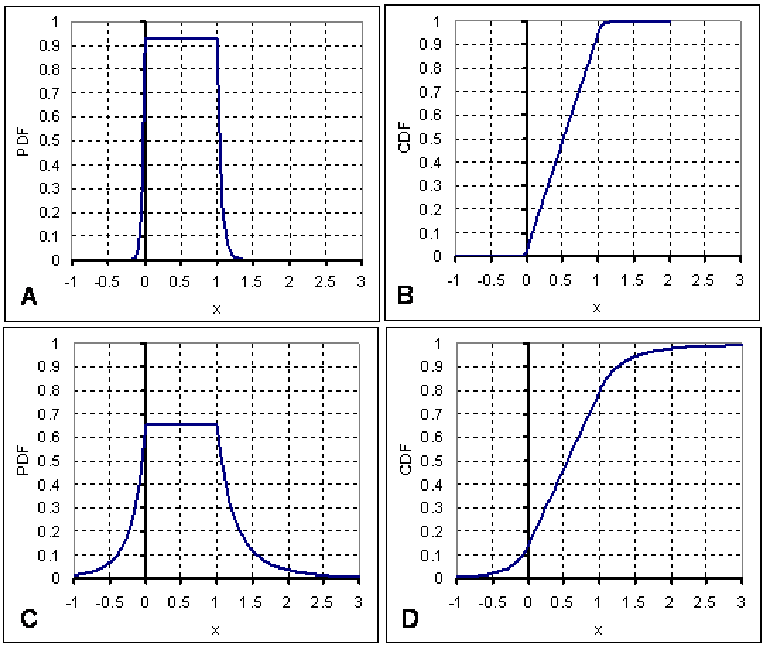

FIGURE 1. Examples of DPU distributions with $m = 37, n = 18.5, \alpha = 0, \beta = 1$ (A-B) and $m = 4.645$, $n = 3.203$, $\alpha = 0$, $\beta = 1$ (C-D). A-C graph the Probability Density Function (2.1) and B-D graph the Cumulative Distribution Function (2.3).

support, but with a range $[0, 1]$ specified for its mode. Both distributions in Figure 1 are asymmetric since the left and right tail behaviors are different.

2.1. Inverse Cumulative Distribution Function

From (2.3) we immediately derive the inverse cdf in a closed form given by:

$$F^{-1}(y|\alpha,\beta,m,n) = \begin{cases} \lambda_1(\beta-\alpha)+\alpha, & 0 \le y < \pi_1, \\ \lambda_2(\beta-\alpha)+\alpha, & \pi_1 \le y \le 1-\pi_3\ , \\ \lambda_3(\beta-\alpha)+\alpha, & 1-\pi_3 \le y \le 1. \end{cases} \tag{2.5}$$

where the weights $\lambda_i, i = 1, \ldots, 3$, are given by

$$\begin{cases} \lambda_1 = 1 - \sqrt[m]{\frac{\pi_1}{y}} \le 0, & 0 \le y < \pi_1, \\ 0 \le \lambda_2 = 1 - \frac{y-\pi_1}{m\pi_1} \le 1, & \pi_1 \le y \le 1-\pi_3, \\ \lambda_3 = \sqrt[n]{\frac{\pi_3}{1-y}} \ge 1, & 1-\pi_3 \le y \le 1. \end{cases}$$

Note that λ_2 does depend on n via the definition of π_1 in (2.2). In addition, every quantile of a DPU distribution is written as linear combination of the central stage bounds α and β in a form reminiscent of the inverse cdf of uniform distribution with support $[\alpha,\beta]$. Sampling from a DPU distribution is straightforward, utilizing the inverse cumulative distribution function theorem and a pseudo-uniform random number generator (see, e.g., Banks et al. (2001)).

2.2. Moments

Setting

$$Y=\frac{X-\alpha}{\beta-\alpha} \tag{2.6}$$

one obtains using the pdf (2.1) the "standardized" DPU pdf of the random variable Y as:

$$f(y|0,1,n,m)=\frac{mn}{m+mn+n}\begin{cases}(1-y)^{-m-1}, & y<0,\\ 1, & 0\leq y\leq 1\ ,\\ y^{-n-1}, & y>1.\end{cases} \tag{2.7}$$

Evidently, standardized DPU distributions have a $uniform[0,1]$ central stage (see Figure 1). The central moments for X follow immediately from those of Y, utilizing the relationship

$$E[(X-E[X])^k]=(\beta-\alpha)^k E[(Y-E[Y])^k]. \tag{2.8}$$

From (2.8) we may conclude that for the DPU distribution the standard statistical measures of variability and shape such as the coefficient of variation (CV), skewness $(\sqrt{\beta_1})$ and kurtosis (β_2) given by $CV=\mu_2/\mu_1^2$, $\sqrt{\beta_1}=sign\{\mu_3\}\sqrt{\mu_3^2/\mu_2^3}$, $\beta_2=\mu_4/\mu_2^2$, where the central moments $\mu_k=E[(Z-E[Z])^k]$, $k=2,3,4$ are functions of solely the tail shape parameters m and n and not of the mode location parameters α and β. (Here Z represents either X or Y).

To calculate the k-th moment of Y around zero we shall consider first the last part of (2.7) and evaluate

$$\int_1^\infty y^{k-n-1}dy=\frac{1}{n-k} \tag{2.9}$$

under the condition $n>k$. For $n\leq k$, the integral (2.9) does not exist. For the second part of (2.7) one obtains:

$$\int_0^1 y^k dy=\frac{1}{k+1}. \tag{2.10}$$

Finally, defining

$$G(k,m+1)=\int_{-\infty}^0\frac{y^k}{(1-y)^{m+1}}dy \tag{2.11}$$

and utilizing integration by parts under the condition that $m>k$ results in

$$G(k,m+1)=-\frac{k}{m}G(k-1,m). \tag{2.12}$$

From (2.11) $G(0, m+1) = 1/m$ and a repeated application of (2.12) yields

$$G(k, m+1) = (-1)^k \left\{ \prod_{j=0}^{k-1} \frac{k-j}{m-j} \right\} \frac{1}{m-k}. \qquad (2.13)$$

With $m > k$ and $n > k$ the k-th moment of the standardized Y ((2.6)) about zero follows from (2.7), (2.9), (2.10) and (2.13) to be:

$$\mu_k' = E[Y^k] = \frac{(-1)^k n(n-k)(k+1)! + n(n+1)\prod_{i=0}^{k}(m-i)}{(k+1)(n-k)(m+mn+n)\prod_{i=1}^{k}(m-i)}. \qquad (2.14)$$

Setting $k = 1$ and $k = 2$ we obtain

$$E[Y] = \frac{-2n(n-1) + n(n+1)m(m-1)}{2(n-1)(m+mn+n)(m-1)},$$

$$E[Y^2] = \frac{3n(n-2) + n(n+1)m(m-1)(m-2)}{3(n-2)(m+mn+n)(m-1)(m-2)}, \qquad (2.15)$$

respectively. In the special symmetric case with $n = m > 0$ (see the pdf (4) with $\alpha = 0$, $\beta = 1$), the mean value $E[Y]$ reduces to $\frac{1}{2}$ (as evident from the symmetry of Y around $\frac{1}{2}$) while substituting $m = n > 0$ in (2.15) yields

$$E[Y^2] = \frac{3 + n(n-1)(n+1)}{3(n-1)(n-2)(n+2)} \to \frac{1}{3} \quad \text{as } m = n \to \infty.$$

Hence, the variance $Var[Y] = E[Y^2] - E^2[Y] \to \frac{1}{12}$ as $m = n \to \infty$ (in accordance with the earlier observation that the $uniform[0,1]$ distribution is the limiting distribution of the pdf (2.7) as $n \to \infty, m \to \infty$). Figure 2A (Figure 2B) below plots the behavior of $E[Y] - \frac{1}{2}$ (of $Var[X] - \frac{1}{12}$) as a function of the tail shape parameters m and n. Note that keeping m fixed (n fixed) and letting the right tail shape parameter $n \downarrow 1$ (the left tail shape parameter $m \downarrow 1$), we obtain $E[Y] \to \infty$ ($E[Y] \to -\infty$). The parameters m and n ought to be larger than 1 for the mean $E[Y]$ to exist. Similarly, observe from Figure 2B that the variance $Var[Y] \to \infty$ when either m or $n \downarrow 2$.

From the conditions $m > k$, $n > k$ for the moments about zero (2.14) it follows that kurtosis $\beta_2 = \mu_4/\mu_2^2$ for DPU distributions takes a finite value when the conditions $m > 4$, $n > 4$ hold. Moreover, kurtoses for DPU distributions may take arbitrarily large values for the values of the parameters m or n arbitrarily close but larger than the value 4 (which explains its leptokurtic behavior). A similar statement can be made regarding the skewness measure except that here we have the condition $m > 3$, $n > 3$ for the skewness to be finite. When $m < n$ ($m > n$) the DPU distribution attains negative (positive) skewness values.

2.3. Limiting Distributions

The mixture probabilities π_1, π_2, π_3 (2.2) are solely functions of the powers m and n and not functions of the location parameters α and β. Keeping $m > 0$ fixed and

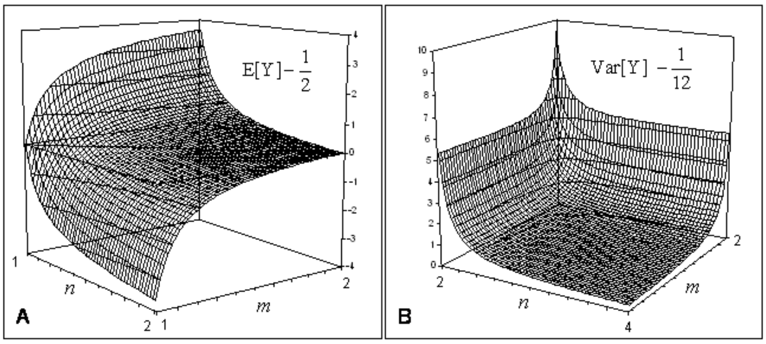

FIGURE 2. $E[Y] - \frac{1}{2}$ (A) and $Var[Y] - \frac{1}{12}$ (B) of a "standardized" DPU pdf (2.7) as a function of the tail shape parameters n and m.

letting $n \to \infty$, one obtains $\pi_1 \to \frac{1}{1+m}$, $\pi_2 \to \frac{m}{1+m}$, $\pi_3 \to 0$ and finally

$$\lim_{n \to \infty} \frac{n(\beta-\alpha)^n}{(x-\alpha)^{n+1}} = \begin{cases} \to 0 & x > \beta \\ \to \infty & x = \beta. \end{cases}$$

Thus in this case, the pdf (2.1) converges to the pdf

$$f(x|\alpha,\beta,m) = \begin{cases} \frac{1}{1+m}\frac{m(\beta-\alpha)^m}{(\beta-x)^{m+1}}, & x < \alpha, \\ \frac{m}{1+m}\frac{1}{\beta-\alpha}, & \alpha \le x \le \beta, \\ 0 & x > \beta, \end{cases} \tag{2.16}$$

which defines a Left-Pareto Uniform (LPU) distribution. Similarly keeping $n > 0$ fixed and letting $m \to \infty$, the pdf (2.1) converges to the Right-Pareto Uniform (RPU) density

$$f(x|\alpha,\beta,n) = \begin{cases} 0, & x < \alpha, \\ \frac{n}{1+n}\frac{1}{\beta-\alpha}, & \alpha \le x \le \beta, \\ \frac{1}{1+n}\frac{n(\beta-\alpha)^n}{(x-\alpha)^{n+1}} & x > \beta. \end{cases} \tag{2.17}$$

Figure 3A (Figure 3B) depicts the limiting LPU distribution (RPU distribution) of the DPU distribution in Figure 1C with parameters $m = 4.645$ ($n = 3.203$), $\alpha = 0, \beta = 1$ (the central part is symmetrical around $\frac{1}{2}$).

Letting $m \to \infty$, $n \to \infty$, the original pdf (2.1) converges to a uniform distribution on $[\alpha, \beta]$. Finally, we have

$$\lim_{\alpha \uparrow \beta} \frac{(\beta-\alpha)^m}{(\beta-x)^{m+1}} = \begin{cases} 0 & x < \alpha \\ \to \infty & x = \alpha \end{cases}$$

for all $m > 0$ and

$$\lim_{\alpha \uparrow \beta} \frac{n(\beta-\alpha)^n}{(x-\alpha)^{n+1}} = \begin{cases} 0 & x > \alpha \\ \to \infty & x = \alpha \end{cases}$$

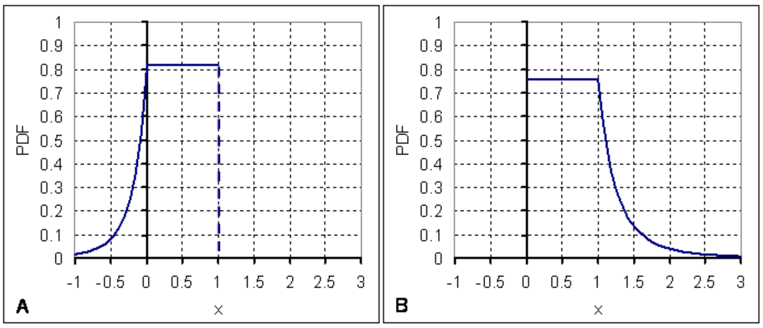

FIGURE 3. A : LPU distribution (2.16) with $m = 4.645$, $\alpha = 0$, $\beta = 1$.
B : RPU distribution (2.17) with $n = 3.203, \alpha = 0$, $\beta = 1$.

for all $n > 0$. Hence, analogously to the uniform distribution with support $[\alpha, \beta]$, the pdf $p(x|\alpha, \beta, m, n)$ (2.1) converges to a single point mass at α (or β) as $\alpha \uparrow \beta$.

3. Elicitation of parameters

Let us first consider the scenario that the boundaries of the central stage α and β are elicited as the p-th and $(1 - r)$-th percentile. Thus here we view the DPU distribution as generalization of the uniform distribution allowing for uncertainty specification beyond its bounds. Denoting a lower quantile a_p and an upper quantile b_r, we set $\alpha = a_p$, $\beta = b_r$ and solve for n and m via (2.3) yielding

$$\begin{cases} \frac{n}{n+mn+m} = p \\ \frac{m}{n+mn+m} = 1 - r \end{cases} \Rightarrow \begin{cases} m = \frac{r-p}{p} \\ n = \frac{r-p}{1-r}. \end{cases} \tag{3.1}$$

Hence, in this case the parameter m (parameter n) is simply the ratio of the probability mass in the central stage to that in the left tail (right tail). Setting $\alpha = a_p = 0$, $p = 0.025$ and $\beta = b_r = 1, r = 0.95$ in (3.1) yields the tail shape parameters $m = 37$ and $n = 18.5$. Figures 1A and 1B plot the DPU pdf and cdf with these parameters.

In the second scenario we assume that the central stage boundary parameters α and β have been elicited as a range for the mode, in addition to a lower percentile $a_p < \alpha$ and upper percentile $b_r > \beta$ to describe uncertainty in both tails. We obtain from (2.3) the following set of non-linear equations (the quantile constraints)

$$\begin{cases} \frac{n}{n+mn+m} \left(\frac{\beta-\alpha}{\beta-a_p}\right)^m = p \\ \frac{m}{n+mn+m} \left(\frac{\beta-\alpha}{b_r-\alpha}\right)^n = 1 - r, \end{cases} \tag{3.2}$$

from which the parameters m and n need to be solved. Rewriting the left hand side (LHS) of the first equation in (3.2) as

$$\psi(m|x,n) = \frac{nx^m}{n+mn+m}$$

where $x = (\beta-\alpha)/(\beta-a_p) \in (0,1)$, it follows that $\psi(m|x,n) \to 0$ ($\psi(m|x,n) \to 1$) as $m \to \infty$ ($m \downarrow 0$), keeping $n > 0$ fixed. In addition,

$$\frac{\partial\psi(m|x,n)}{\partial m} = \{(n+mn+m)ln(x)-(n+1)\}\frac{nx^m}{(n+mn+m)^2} < 0.$$

Hence, the first equation in (3.2) has a unique solution m^* for every fixed value of $n > 0$ and thus it defines an implicit continous function $\xi(n)$ such that the parameter combination $\{m^* = \xi(n),\, \alpha, \beta, n\}$ satisfies the first quantile constraint for all $n > 0$. The unique solution m^* may be solved for by employing a standard root finding algorithm such as, for example, GoalSeek in MicroSoft Excel. Analagously, the second equation defines an implicit continuous function $\zeta(m)$ such that the parameter combination $(m,\, \alpha, \beta, n^* = \zeta(m))$ satisfies the second quantile constraint for all $m > 0$. We propose the following direct algorithm solving (3.2):

Step 1: Set $n^* = 1$.
Step 2: Calculate $m^* = \xi(n^*)$ (satisfying for the first constraint in (3.2)).
Step 3: Calculate $n^* = \zeta(m^*)$ (satisfying for the second constraint in (3.2)).
Step 4: If $\left|\frac{n^*}{n^*+m^*n^*+m^*}\left(\frac{\beta-\alpha}{\beta-a_p}\right)^{m^*} - p\right| < \epsilon$ Then Stop Else Goto Step 2.

Setting $\alpha = 0$, $\beta = 1$, $a_p = -\frac{1}{4}$, $p = 0.05$ and $b_r = 1\frac{1}{4}, r = 0.90$ in (3.2) yields the tail shape $m = 4.645$ and $n = 3.203$. Figures 1C and 1D plot the DPU pdf and cdf possessing these parameters. (A MicroSoft Excel spreadsheet with an implementation of the above algorithm is available from the authors upon request.)

4. Maximum Likelihood Estimation

For a random sample $X = (X_1, \ldots, X_s)$ of size s from the distribution (2.1) the likelihood function is, by definition,

$$L(|\alpha,\beta,m,n) = \left\{\frac{mn}{m+mn+n}\right\}^s \left\{\frac{1}{\beta-\alpha}\right\}^s \times \left\{\prod_{i=1}^{r_1}\frac{\beta-\alpha}{\beta-X_{(i)}}\right\}^{m+1}\left\{\prod_{j=r_2+1}^{s}\frac{\beta-\alpha}{X_{(j)}-\alpha}\right\}^{n+1} \tag{4.1}$$

where $X_{(1)} < X_{(2)} < \ldots < X_{(s)}$ are the order statistics of X, r_1 and r_2 are such that:

$$X_{(r_1)} \le \alpha < X_{(r_1+1)},\; X_{(r_2)} \le \beta < X_{(r_2+1)},\; 0 \le r_1 \le r_2 \le s.$$

By convention

$$X_{(0)} = -\infty,\; X_{(s+1)} = +\infty.$$

Note that, since $\alpha < \beta$ and for the case that

$$r_1 = r_2 = r. \tag{4.2}$$

we have the restriction

$$X_{(r)} \leq \alpha < \beta \leq X_{(r+1)},\ r = 0, \ldots, s.$$

Scenarios with condition (4.2) corresponds to a set of order statistics such that no observations have been obtained in the central stage of the DPU distribution. The following direct algorithm to maximize the likelihood $L(X|\alpha, \beta, m, n)$ [(4.1)] and to calculate the ML estimates of the parameters α, β, m, and n is suggested:

The *k-th* Iteration:
Step 0: Set $k = 1$, $m_1 = 1$, $\alpha_1 = X_{(\lfloor s/3 \rfloor)}$, $\beta_1 = X_{(\lfloor 2s/3 \rfloor)}, n_1 = 1$.
Step 1: Determine n_{k+1} by maximizing $L(X|\alpha_k, \beta_k, m_k, n)$ over n.
Step 2: Determine m_{k+1} by maximizing $L(X|\alpha_k, \beta_k, m, n_{k+1})$ over m.
Step 3: Determine α_{k+1} by maximizing $L(X|\alpha, \beta_k, m_{k+1}, n_{k+1})$ over α.
Step 4: Determine β_{k+1} by maximizing $L(X|\alpha_{k+1}, \beta, m_{k+1}, n_{k+1})$ over β.
Step 5: If $|L(X|\alpha_k, \beta_k, n_k, m_k) - L(X|\alpha_{k+1}, \beta_{k+1}, m_{k+1}, n_{k+1})| < \epsilon$ *STOP*
Else $k = k + 1$ and Goto Step 1.

In Step 0 above we initialize α_1 and β_1 so that each part of the DPU pdf initially contains approximately the same number of observations. Figures 4A-D plot an example of likelihood profiles of (4.1) as a function of m, n, α and β, respectively, for the illustrative data set of size $s = 8$:

$$X = (0.10, 0.25,\ 0.30,\ 0.40,\ 0.45,\ 0.60,\ 0.75,\ 0.80). \tag{4.3}$$

Observe that all likelihood profiles in Figure 4 attain a unique maximum. Note that those profiles of (4.1) as a function of α and β are continuous, but only piecewise differentiable. Figures 4A-C plot the likelihood profiles for n, m, α for the first iteration ($k = 1$, $\alpha_1 = X_{(\lfloor 8/3 \rfloor)} = X_{(2)}$, $\beta_1 = X_{(\lfloor 16/3 \rfloor)} = X_{(5)}$). Figure 4D shows that the unique maximum of the likelihood profile as function of β (or α) in the third iteration ($k = 3$) does not have to be attained at specific order statistic. For the data set (4.3) after 8 iterations we arrive at $\alpha_9 = X_{(2)}$, $\beta_9 = X_{(8)}, m_9 = 4.773$ and $n_9 \rightarrow \infty$. Hence, the resulting ML fit for the data set (4.3) coincides with an LPU distribution (see Figure 3A). Below we shall provide some mathematical details regarding the execution of Step 1-4 of the algorithm above. (A software program with the ML algorithm above (requiring a set of order statistics in an ASCII text file as input) is available from the authors upon request.)

STEP 1: We shall consider separately the cases $\beta < X_{(s)}$ and $\beta = X_{(s)}$.

Case A: $\beta < X_{(s)}$; From $X_{(r_2)} \leq \beta < X_{(r_2+1)}$ it follows that $r_2 \leq s - 1$. Viewing the likelihood profile of (4.1) as a function of n (see Figure 4A for the data set

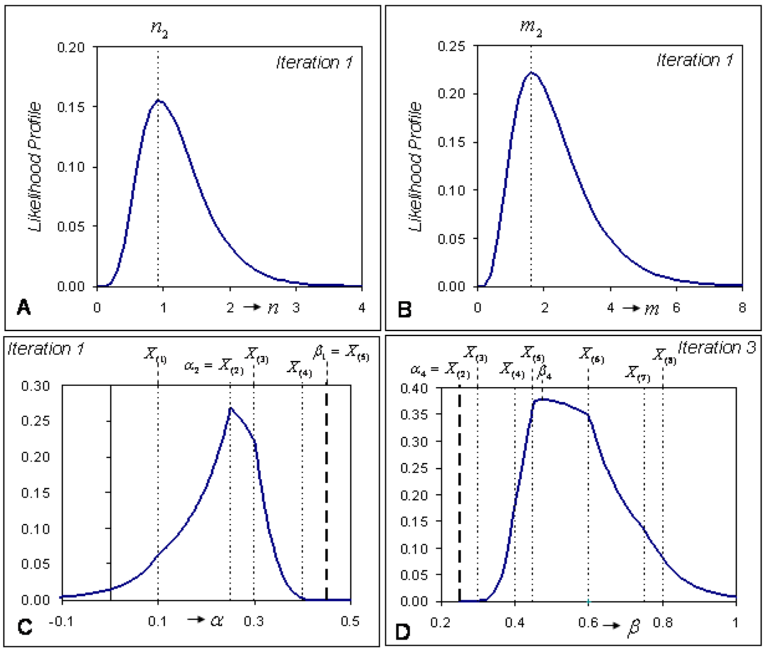

FIGURE 4. Example Likelihood Profiles of (4.1) for (A) the right tail shape parameter n,(B) the left tail shape parameter n, (C) the left mode location parameter α, (D) the right mode location parameter β, for the illustrative data set (4.3).

(4.3)) one can write:

$$L(X|n) \propto \left\{ \frac{mn}{m+mn+n} \right\}^{s} \left\{ \prod_{j=r_2+1}^{s} \frac{\beta-\alpha}{X_{(j)}-\alpha} \right\}^{n+1} . \tag{4.4}$$

Maximizing $L(X|n)$ given by (4.4) is equivalent to maximizing its logarithm

$$Log\{L(X|n)\} = sLog\left\{ \frac{mn}{m+mn+n} \right\} + (n+1)Log\left\{ \prod_{j=r_2+1}^{s} \frac{\beta-\alpha}{X_{(j)}-\alpha} \right\} + \mathcal{C} \tag{4.5}$$

where $\mathcal{C}$ is a constant. Setting the partial derivative with respect to n in (4.5) equal to zero, the following quadratic equation in the parameter n is obtained

$$(m+1)n^2 + mn + mLog^{-1}\left\{ \sqrt[s]{\prod_{j=r_2+1}^{s} \frac{\beta-\alpha}{X_{(j)}-\alpha}} \right\} = 0. \tag{4.6}$$

Since $n > 0$, solving (4.6) for n it follows immediately that

$$n^{k+1} = -\frac{m}{2(m+1)} + \frac{1}{2(m+1)}\sqrt{m^2 - 4(m+1)Log^{-1}\left\{\sqrt[s]{\prod_{j=r_2+1}^{s}\frac{\beta-\alpha}{X_{(j)}-\alpha}}\right\}}\ . \tag{4.7}$$

Case B: $\beta = X_{(s)}$; Since $X_{(r_2)} \leq \beta < X_{(r_2+1)}$ we have that $r_2 = s$. Considering the likelihood profile of (4.1) as a function of n we obtain

$$L(X|n) \propto \left\{\frac{mn}{m+mn+n}\right\}^s. \tag{4.8}$$

Maximizing the logarithm of the RHS of (4.8) and taking its derivative with respect to n yields the positive value

$$\frac{1}{n}\frac{m}{m+mn+n} > 0.$$

Hence, the RHS of (4.8) is a strictly increasing function in n and $n_k \to \infty$. Thus the maximum likelihood estimation of a DPU distribution (2.1), reduces to the same estimation of a LPU distribution (2.16) (see also Figure 3A), which could be achieved by setting n to be an arbitrarily fixed large value in the algorithm presented above.

STEP 2**:** As above we shall consider separately the cases $\alpha > X_{(1)}$ and $\alpha = X_{(1)}$.

Case A: $\alpha > X_{(1)}$; The likelihood profile of (4.1) with $\alpha > X_{(1)}$ as a function of m for the data set (4.3) is plotted in Figure 4B. From $X_{(r_1)} \leq \alpha < X_{(r_1+1)}$ we have that $r_1 \geq 2$. Analogously to Step 1, it follows that

$$m^{k+1} = -\frac{n}{2(n+1)} + \frac{1}{2(n+1)}\sqrt{n^2 - 4(n+1)Log^{-1}\left\{\sqrt[s]{\prod_{i=1}^{r_1}\frac{\beta-\alpha}{\beta-X_{(i)}}}\right\}}\ . \tag{4.9}$$

(Compare with (4.7).)

Case B: $\alpha = X_{(1)}$; From $X_{(r_1)} < \alpha \leq X_{(r_1+1)}$ we have that $r_1 = 1$. Here maximum likelihood estimation of a DPU distribution (2.1), reduces to a maximum likelihood estimation of a RPU distribution (2.17) (see also Figure 3B), which could be achieved by setting m to be an arbitrarily large value in the algorithm above.

STEP 3: Considering the likelihood profile of (4.1) as a function of α (see Figure 4C for the data set (4.3)) we have

$$L(X|\alpha) \propto (\beta-\alpha)^{\xi} \times \left\{ \prod_{j=r_2+1}^{s} \frac{1}{X_{(j)}-\alpha} \right\}^{n+1}, \tag{4.10}$$

where

$$\xi = r_1(m+1) + (s-r_2)(n+1) - s \tag{4.11}$$

and $X_{(r_1)} \leq \alpha < X_{(r_1+1)} < \beta$, $0 \leq r_1 < r_2$, or $X_{(r_1)} \leq \alpha < \beta, 0 \leq r_1 = r_2$. Maximizing the logarithm of (4.10) is equivalent to maximizing

$$\frac{\xi}{n+1} Log(\beta-\alpha) - \sum_{j=r_2+1}^{s} Log(X_{(j)}-\alpha). \tag{4.12}$$

Consider the two cases $\xi \leq 0$ and $\xi > 0$.

Case A: $\xi \leq 0$; From functions of the form $Log(\theta-\alpha)$ being strictly decreasing functions for $\alpha < \theta, \xi \leq 0$ and $n > 0$ it immediately follows noting the restrictions

$$\alpha < X_{(j)},\ j = r_2+1, \ldots, s \text{ and } \alpha < \beta,$$

that (4.12) is a strictly increasing function for $\alpha \in [X_{(r_1)}, \mathcal{U}]$, where

$$\mathcal{U} = \begin{cases} X_{(r_1+1)} & r_1 \in \{0, \ldots, r_2-1\} \\ \beta & r_1 = r_{2.} \end{cases} \tag{4.13}$$

Hence, (4.12) attains is maximum at the upper bound of the range $[X_{(r_1)}, \mathcal{U}]$.

Case B: $\xi > 0$; Taking the derivative of (4.12) with respect to α and equating it to zero yields

$$\sum_{j=r_2+1}^{s} \frac{\beta-\alpha}{X_{(j)}-\alpha} = \frac{\xi}{n+1}. \tag{4.14}$$

Taking the derivative of the LHS of (4.14) with respect to α we obtain

$$\sum_{j=r_2+1}^{s} \frac{\beta - X_{(j)}}{(X_{(j)}-\alpha)^2}.$$

Hence, from $\beta \leq X_{(r_2+1)}$, $\beta < X_{(j)}$, $j \in \{r_2+1, \ldots, s\}$ it follows that LHS of (4.14) is a strictly decreasing function in α over the range $[X_{(r_1)}, \mathcal{U}]$, where $\mathcal{U}$ is given in (4.13). We are now able to conclude from (4.14) and the definition of ξ (4.11), that whenever $\xi > 0$ and:

1. $\sum_{j=r_2+1}^{s} \frac{\beta-\mathcal{U}}{X_{(j)}-\mathcal{U}} > \frac{\xi}{n+1} \Rightarrow$ (4.10) attains it maximum at $\mathcal{U}$.
2. $\sum_{j=r_2+1}^{s} \frac{\beta-X_{(r_1)}}{X_{(j)}-X_{(r_1)}} < \frac{\xi}{n+1} \Rightarrow$(4.10) attains it maximum at $X_{(r_1)}$.
3. $\sum_{j=r_2+1}^{s} \frac{\beta-\mathcal{U}}{X_{(j)}-\mathcal{U}} < \frac{\xi}{n+1} < \sum_{j=r_2+1}^{s} \frac{\beta-X_{(r_1)}}{X_{(j)}-X_{(r_1)}} \Rightarrow$(4.10) attains it maximum at a stationary point $\alpha^* \in [X_{(r_1)}, \mathcal{U}]$.

STEP 4 - See Figure 4D: Considering the likelihood profile of (4.1) as a function of β (see Figure 4D for the data set (4.3)) we have

$$L(X|\beta) \propto (\beta-\alpha)^{\xi} \times \left\{ \prod_{i=1}^{r_1} \frac{1}{\beta - X_{(i)}} \right\}^{m+1} \tag{4.15}$$

where as before ξ is defined by Eq. (4.11) and $\alpha < X_{(r_2)} < \beta \leq X_{(r_2+1)}, r_1 < r_2 \leq s+1$, or $\alpha < \beta \leq X_{(r_2+1)}, r_1 = r_2 \leq s+1$. Maximizing the logarithm of (4.15) is equivalent to maximizing

$$\frac{\xi}{m+1} Log(\beta-\alpha) - \sum_{i=1}^{r_1} Log(\beta - X_{(i)}). \tag{4.16}$$

Consider the two cases $\xi \leq 0$ and $\xi > 0$.

Case A: $\xi \leq 0$; From functions of the form $Log(\beta-\theta)$ being strictly increasing functions for $\theta < \beta, \xi \leq 0$ and $m > 0$ it immediately follows for

$$X_{(r_2)} < \beta,\ i = 1, \ldots, r_1 \text{ and } \alpha < \beta,$$

that (4.16) is a strictly decreasing function for $\beta \in [\mathcal{L}, X_{(r_2+1)}]$, where

$$\mathcal{L} = \begin{cases} X_{(r_2)} & r_2 \in \{r_1+1, \ldots, s\} \\ \alpha & r_1 = r_{2.} \end{cases} \tag{4.17}$$

Hence, (4.16) attains is maximum at the lower bound of the range $[\mathcal{L}, X_{(r_2+1)}]$.

Case B: $\xi > 0$; Taking the derivative of (4.16) with respect to β and equating it to zero yields

$$\frac{\xi}{m+1} = \sum_{i=1}^{r_1} \frac{\beta-\alpha}{\beta - X_{(i)}}. \tag{4.18}$$

Taking the derivative of the RHS of (4.18) with respect to β yields

$$\sum_{i=1}^{r_1} \frac{\alpha - X_{(i)}}{(\beta - X_{(i)})^2}.$$

Hence, from $\alpha \geq X_{(r_1)}$, $\alpha > X_{(i)}$, $i \in \{1, \ldots, r_1 - 1\}$ it follows that RHS of (4.18) is a strictly increasing function in β over the range $[\mathcal{L}, X_{(r_2+1)}]$, where $\mathcal{L}$ is given by (4.17). We are now able to conclude from (4.18) and the definition of ξ (4.11), that whenever $\xi > 0$ and:

1. $\frac{\xi}{m+1} < \sum_{i=1}^{r_1} \frac{\mathcal{L}-\alpha}{\mathcal{L}-X_{(i)}} \Rightarrow$(4.15) attains it maximum at $\mathcal{L}$.
2. $\frac{\xi}{m+1} > \sum_{i=1}^{r_1} \frac{X_{(r_2+1)}-\alpha}{X_{(r_2+1)}-X_{(i)}} \Rightarrow$(4.15) attains it maximum at $X_{(r_2+1)}$.
3. $\sum_{i=1}^{r_1} \frac{\mathcal{L}-\alpha}{\mathcal{L}-X_{(i)}} < \frac{\xi}{m+1} < \sum_{i=1}^{r_1} \frac{X_{(r_2+1)}-\alpha}{X_{(r_2+1)}-X_{(i)}} \Rightarrow$ (4.15) attains its maximum at a stationary point $\alpha^* \in [\mathcal{L}, X_{(r_2+1)}]$.

5. An Example using Financial Data

We shall illustrate the ML procedure for the DPU distribution utilizing the monthly US certificate deposit (CD) interest rate data for the period from 1964-2004. Our aim is to construct a realization of a time series $\nu_k, k = 0, 1, 2, \ldots$, from this data, where the ν_k are i.i.d. random variables. This would provide us with an i.i.d. sample for our ML procedure. To construct such a realization we shall use the (by now standard) Auto-Regressive Conditional Heteroscedastic (ARCH) time series model devised by the 2003 Nobel Laureate in Economics R.F. Engle in 1982.

Denoting the interest rate after month k by i_k, we shall begin by assuming a simple financial engineering models for the random behavior of the interest rate, namely the multiplicative model

$$i_{k+l} = i_k \cdot \epsilon_{k,l} \Leftrightarrow Ln(\epsilon_{k,l}) = Ln(i_{k+l}) - Ln(i_k) \tag{5.1}$$

where $k = 1, 2, \ldots$, the gap $l = 1, 2, \ldots$ and and $\epsilon_{k,l}$ are mutually independent random variables (see, e.g., Leunberger (1998)). (Our interest rate i_1 is the CD rate in June of 1964.) Setting the gap to be $l = 3$ we obtain 162 data points and the values of the auto-correlation function

$$ACF(\lambda, 1) = Corr[Ln(\epsilon_{k+\lambda,3}), Ln(\epsilon_{k,3})]$$

with lags $\lambda = 1, \ldots, 5$ provided in Table 1. Table 1 also contains the values of the Ljung-Box Q statistics $[LBQ(\lambda)]$ (see Ljung and Box (1978)) and their p-values for testing the null hypothesis that the auto-correlations for all lags up to the lag λ equal zero. Tsay (2002) recommends that $\lambda \approx Ln(162) = 5.08$ performs as the best (in terms of statistical power) value. Note that the p-values associated with the three step differences $Ln(\epsilon_{k,3})$ indicate that we fail to reject the null hypothesis (i.e. the auto-correlations for all lags up to lag λ equal zero) for all $\lambda = 1, \ldots, 5$. Hence, we are justified to conclude that the time series $Ln(\epsilon_{k,3})$ defined by (5.1) is serially uncorrelated.

We next standardize the time series $Ln(\epsilon_{k,3})$ utilizing the systematic framework for volatility modeling provided by the same Engle's (1982) ARCH model. Specifically, an $ARCH(p)$ model assumes that

$$a_k = \sigma_k \nu_k,\ \sigma_k^2 = \alpha_0 + \alpha_1 a_{k-1}^2 + \ldots + \alpha_p a_{k-p}^2 \tag{5.2}$$

where α_i , $i = 0, \ldots, m,$ are constants, a_k are serially uncorrelated and ν_k is a sequence of i.i.d. random variables with zero mean and variance of one. For our data involving $Ln(\epsilon_{k,3})$, we have

$$\overline{Ln(\epsilon_{k,3})} = \frac{1}{160}\sum_{k=1}^{160} Ln(\epsilon_{k,3}) = -0.0028; \tag{5.3}$$

$$s^2 = \frac{1}{159}\sum_{k=1}^{160}(Ln(\epsilon_{k,3}) - \overline{Ln(\epsilon_{k,3})}) = 2.456e - 2.$$

TABLE 1. Auto-Correlation Function, Ljung-Box Q Statistic and p-values for three step log-differences $Ln(\epsilon_{k,3})$ with Lags 1, ..., 5. Partial Auto-Correlation Function of a_k^2 (cf. (5.4)) with Lags 1,..., 5.

	Three-Step Log Differences			$(a_k)^2$	
Lag	ACF	LBQ	p-value	PACF	t-Statistic
1	0.115	2.185	0.1394	0.250	3.176
2	-0.027	2.309	0.3152	0.255	3.250
3	0.158	6.457	0.0914	0.001	0.007
4	0.053	6.927	0.1398	-0.106	-1.353
5	0.062	7.585	0.1806	-0.062	-0.788

(here s^2 is the sample variance estimator of $Ln(\epsilon_{k,3})$, $k = 1, \ldots, 160$). Hence the time series

$$a_k = \frac{Ln(\epsilon_{k,3})}{s} \tag{5.4}$$

may be considered as a realization of (5.2). It would thus follow that using the estimates (5.3) and rescaling $Ln(\epsilon_{k,2})$ as in (5.4), one achieves the conditions of a zero mean and variance 1 of ν_k in (5.2).

From the observation that a_k being serially uncorrelated (in view of (5.4) and the fact that $Ln(\epsilon_{k,3})$ are serially uncorrelated) and the partial auto-correlation function (PACF) values of a_k^2 presented in Table 1, it follows that the time series a_k does not exhibit a constant variance (i.e. it is heteroscedastic), but could be appropriately represented by an $ARCH(2)$ model (see, the second row in Table 1 and Tsay (2002) for a more detailed explanation). We thus obtain the following equation for σ_k^2:

$$\sigma_k^2 = 0.571 + 0.186a_{k-1}^2 + 0.258a_{k-1}^2, \tag{5.5}$$

where the parameters $\alpha = (\alpha_0, \alpha_1, \alpha_2) \approx (0.571, 0.186, 0.258)$ are estimated using the least squares method (see (5.2)). The analysis in Table 2 involving $\nu_k = a_k/\sigma_k$ suggests that now the time series ν_k is serially uncorrelated and homoscedastic. From (5.4), (5.5) and (5.2) it follows that the time series ν_k may be interpreted as that of standardized quarterly log-differences of monthly US CD interest rates from 1964-2004 (where as mention above i_1 is the monthly CD interest rate in June 1964).

The empirical pdf of the times series ν_k is depicted in Figure 5 together with the ML fitted Gaussian pdf (Figure 5A) with parameters μ and σ, asymmetric Laplace (AL) pdf (Figure 5B), the later given by

$$f(x|\theta, \kappa, \sigma) = \begin{cases} \frac{\sqrt{2}}{\sigma}\frac{\kappa}{1+\kappa^2} exp\left[-\sqrt{2}\,\frac{1}{\sigma\kappa}(\theta - x)\right], & \text{for } x < \theta \\ \frac{\sqrt{2}}{\sigma}\frac{\kappa}{1+\kappa^2} exp\left[-\sqrt{2}\,\frac{\kappa}{\sigma}(x - \theta)\right], & \text{for } x \geq \theta, \end{cases} \tag{5.6}$$

TABLE 2. Auto-Correlation Function, Ljung-Box Q Statistic and p-values for three step log-differences $Ln(\epsilon_{k,3})$ with Lags 1,..., 5. Partial Auto-Correlation Function of a_k^2 (cf. (5.4)) with Lags 1,..., 5.

	ν_k			$(\nu_k)^2$	
Lag	ACF	LBQ	p-value	PACF	t-Statistic
1	0.196	6.288	0.0122	0.080	1.017
2	0.060	6.883	0.0320	0.003	0.039
3	0.094	8.341	0.0395	0.105	1.328
4	0.073	9.218	0.0559	-0.094	-1.192
5	0.061	9.832	0.0801	-0.064	-0.806

TABLE 3. Maximum likelihood parameter estimates for the theoretical distributions depicted in Figure 5.

Gaussian (Fig. 5A)	$\hat{\mu} = -3.20e - 2$	$\hat{\sigma} = 0.981$		
AS Laplace (Fig. 5B)	$\hat{\theta} = 0.113$	$\hat{\sigma} = 1.016$	$\hat{\kappa} = 1.106$	
TSP (Fig. 5C)	$\hat{a} = -7.645$	$\hat{\theta} = 0.257$	$\hat{b} = 5.466$	$\hat{n} = 8.118$
DPU (Fig. 5D)	$\hat{m} = 2.133$	$\hat{\alpha} = -0.429$	$\hat{\beta} = 0.788$	$\hat{n} = 3.324$

where $\kappa, \sigma > 0,$ Two-Sided Power (TSP) pdf (Figure 5C) given by

$$f(x|a,\theta,b,n) = \begin{cases} \frac{n}{(b-a)}\left(\frac{x-a}{\theta-a}\right)^{n-1}, & \text{for } a < x \leq \theta \\ \frac{n}{(b-a)}\left(\frac{b-x}{b-\theta}\right)^{n-1}, & \text{for } \theta < x < b, \end{cases} \tag{5.7}$$

where $a \leq m \leq b$, $n > 0$ (see, Kotz and van Dorp (2004)) and DPU pdf (2.1) (Figure 5D). (For ML estimators of the Gaussian (normal) parameters see a standard text in statistics, e.g., Mood et al., 1974.) Kotz et. al (2001) and Kotz and van Dorp (2004) discuss a ML procedure for the asymmetric Laplace and TSP distributions, respectively. Table 3 contains the ML estimates of the parameters of the pdf's in Figure 5. From Figure 5 we can observe (by a careful visual comparison) that the DPU distribution (Figure 5D) provides a "better" fit to the empirical cdf amongst these competing distributions (at least at the central stage).

A formal fit analysis is conducted in Table 4, where the Chi-square statistic

$$\sum_{i=1}^{16} \frac{(O_i - E_i)^2}{E_i} \tag{5.8}$$

is calculated utilizing 13 bins ($13 \in [\sqrt{160}, 160/5]$ as suggested by Banks et al. 2001). The upper boundaries UB_i of the bins are selected in the manner that the expected number of observations E_i, $i = 1, \ldots, 13,$ in each Bin i equal $160/13$ and are presented in Table 4. Such a boundary selection procedure follows the "equal probability method of constructing classess" (see, e.g., Stuart and Ord (1994)), resulting in different boundaries for each one of the four distributions DPU, TSP,

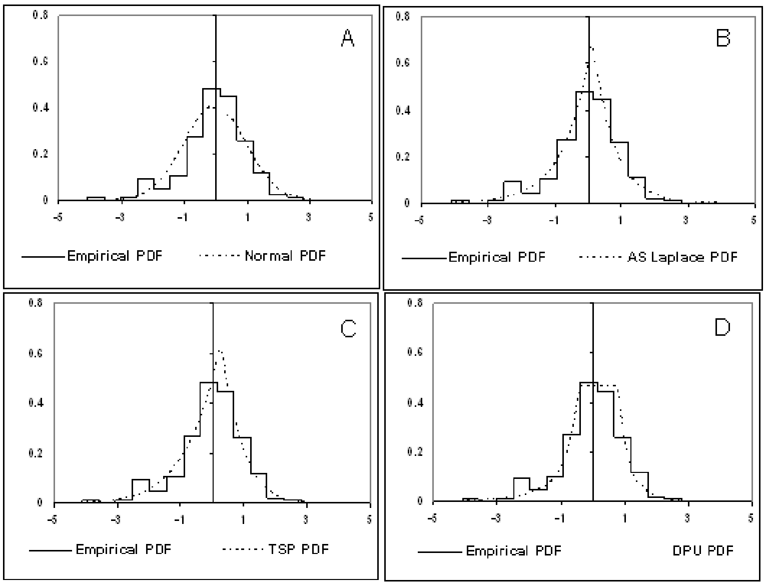

FIGURE 5. Empirical pdf's of standardized quarterly log-differences of monthly US CD interest rates from 1964-2004 together with ML fitted theoretical distributions. Maximum likelihood parameter estimates are provided in Table 3.

AL and Gaussian (Normal) in Table 4. The values O_i in (5.8) are the actual number of observations in each Bin i. While all of the four distributions appear to be appropriate (since we fail to reject the null hypothetis based on the p-values of the Chi-squared test), the Gaussian distribution evidently produces the worst fit with a p-value of 39%. The DPU distribution results in the largest p-value (73%) of the corresponding chi-squared test. In the authors' opinion the behavior of the fit analysis as given in Table 4 and Figure 4 justifies the conclusion about the suitability of the DPU distribution for the data under consideration.

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Table 4. Goodness of Fit Analysis of MLE fitted distributions for the 1964-2004 data on quarterly log-differences of monthly US CD interest rates.

	DPU	TSP	AL	Normal	DPU	TSP	AL	Normal
Bin i	UB_i	UB_i	UB_i	UB_i	$\frac{(O_i-E_i)^2}{E_i}$	$\frac{(O_i-E_i)^2}{E_i}$	$\frac{(O_i-E_i)^2}{E_i}$	$\frac{(O_i-E_i)^2}{E_i}$
1	-1.39	-1.51	-1.45	-1.43	0.233	0.233	0.233	0.233
2	-0.78	-0.97	-0.90	-1.03	0.233	0.889	0.139	2.289
3	-0.51	-0.63	-0.58	-0.75	0.008	0.008	0.039	2.289
4	-0.34	-0.37	-0.35	-0.52	0.889	0.008	0.139	0.139
5	-0.17	-0.17	-0.17	-0.32	0.233	1.108	0.233	0.008
6	-0.01	0.00	-0.03	0.13	0.008	0.008	0.433	0.589
7	0.16	0.15	0.10	0.06	0.233	0.008	0.889	0.039
8	0.33	0.28	0.21	0.26	0.039	0.008	0.433	1.108
9	0.49	0.42	0.36	0.46	0.039	0.139	0.039	1.789
10	0.66	0.59	0.55	0.69	2.289	1.508	0.139	0.233
11	0.83	0.83	0.81	0.97	0.039	2.633	3.639	0.589
12	1.12	1.21	1.26	1.37	0.433	0.139	0.039	0.889
13	>1.12	>1.21	>1.26	>1.37	0.589	0.039	0.008	0.433
Chi-Squared Statistic					5.26	6.73	6.40	10.63
Degrees of Freedom					8	8	9	10
p-value					0.73	0.57	0.70	0.39

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