A Versatile Bivariate Distribution on a Bounded Domain: Another Look at the Product Moment Correlation

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Abstract - The Fairlie-Gumbel-Morgenstern (FGM) family has been investigated in detail for various continuous marginals such as Cauchy, normal, exponential, gamma, Weibull, lognormal and others. It has been a popular model for the bivariate distribution with mild dependence. However, bivariate FGM's with continuous marginals on a bounded support discussed in the literature are only those with uniform or power marginals. In this note we study the bivariate FGM family with marginals given by the recently proposed two-sided power (TSP) distribution. Since this family of bounded continuous distributions is very flexible, the properties of the FGM family with TSP marginals could serve a reliable indication of the structure of the FGM distribution with arbitrary marginals defined on a compact set. A remarkable stability of the correlation between the marginals has been observed.

1. INTRODUCTION

The arsenal for constructing continuous bivariate distribution with specified marginals on a bounded set is rather limited. Most continuous bivariate distributions with specified marginals are constructed on either the whole $\mathbb{R}^2$ or on a positive orthant. This is due mainly to the fact that the vast majority of univariate continuous distributions are defined on an infinite interval (with

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the notable exceptions of the beta distribution - and its particular cases uniform and power distributions - and the triangular distributions.) Indeed, the by far most widely used continuous distribution on a bounded interval is the uniform followed by the beta distribution, the power distribution and only recently some attention has began to be paid to the triangular distribution (see, e.g., Johnson (1997) and Johnson and Kotz (1999)). The fact that bivariate distributions with specified continuous marginals have not been investigated for the case of beta marginals may partially be explained by the fact that specification of marginals requires the use of the cumulative distribution function (cdf) in a closed form, not available for this distribution. Actually, to the best of our knowledge, even bivariate distributions with specified triangular marginals have not been studied. While the Dirichlet and Ordered Dirichlet distribution are multivariate distributions with beta marginals (on a bounded support), they do not allow for arbitrary specification of beta marginals (see e.g. Kotz et al. (2000) and Van Dorp et al. (1996)). It would seem that construction of bivariate distributions with arbitrary specified marginals with bounded support may still be considered an unchartered area.

Fortunately there exist several mechanisms for constructing bivariate distributions. A new method of generating bivariate distribution based on the modification of a bivariate uniform distribution and a subsequent "cloning" has been proposed by Johnson and Kotz (2001). Earlier mechanisms for constructing bivariate distributions with arbitrary specified marginals center around the use of copulas (see, e.g., Genest and Mackay (1986)). The easiest and most natural approach embodied in the general class of bivariate distributions is perhaps the one originally proposed by Morgenstern (1956), Gumbel (1960) and Farlie (1960) known in the literature as the FGM family. This family has only one governing parameter which determines the degree of dependence (in addition to the parameters of the marginal distributions). A vast literature on this topic exists (see, e.g., by now the recent book by Drouet and Kotz (2001)).

The starting point for the constructing of a FGM family is the case of uniform marginals. Here the bivariate cdf is given by

$$F_{U_1,U_2}(u_1,u_2) = u_1 u_2[1+\alpha(1-u_1)(1-u_2)],\ |\alpha|<1,\ 0\le u_i\le 1, i=1,2. \tag{1}$$

and the pdf is simply

$$f_{U_1,U_2}(u_1,u_2) = [1+\alpha(1-2u_1)(1-2u_2)],\ |\alpha|<1,\ 0\le u_i\le 1, i=1,2. \tag{2}$$

In general a FGM bivariate distribution with marginal cdf's $F_{X_1}(x_1),\ F_{X_2}(x_2)$ and pdf's $f_{X_1}(x_1), f_{X_2}(x_2),$ is represented as

$$F_{X_1,X_2}(x_1,x_2) = F_{X_1}(x_1)F_{X_2}(x_2)[1+\alpha(1-F_{X_1}(x_1))(1-F_{X_2}(x_2))], \tag{3}$$

or in terms of pdf's (if they exist) as

$$f_{X_1,X_2}(x_1,x_2) = f_{X_1}(x_1)f_{X_2}(x_2)[1+\alpha(1-2F_{X_1}(x_1))(1-2F_{X_2}(x_2))], \tag{4}$$

where as above $|\alpha|<1,\ 0\le u_i\le 1, i=1,2$. The basic theory of FGM distributions and its correlation structure has been investigated for various continuous marginals with unbounded support such as Cauchy, Normal, exponential, gamma, Weibull, lognormal and many others in Johnson and Kotz (1975), (1977), Schucany et al. (1978), Cook and Johnson (1986), D'Este (1981), Lingappaiah (1984), Barnett (1965) and Drouet and Kotz (2001), among other sources. As to FGM distributions with bounded support, besides those with uniform marginals, only the case of power marginals with the density $f_X(x) = nx^{n-1},\ 0\le x\le 1,\ n>0$ seem to receive any attention. (See, e.g., Schucany et al. (1978)).

The product moment correlation coefficient for <u>all</u> the FGM distributions with continuous marginals is less than $1/3$ with the maximum being attained for the uniform marginals. The model is thus applicable to the cases of a "mild" correlation which occurs in engineering and medical applications (see, e.g., Blischke and Prabhaker Murthy (2000) and Chalabian and Dunnington (1998)). The maximum correlation coefficient for normal marginals is $1/\pi$, $1/4$ for exponential marginals, 0.281 for the Laplace and gamma marginals (with the shape parameter equal to 2) and is $n(n+2)/(2n+1)^2$ (ranging from 0 to $1/3$) for power marginals.

We shall study the correlation structure of a bivariate FGM family with marginals given by the two parameter, two-sided power (TSP) distribution introduced by van Dorp and Kotz (2001a,b). The distribution in its canonical form is given by the cdf

$$F_X(x) = \begin{cases} \theta\left(\frac{x}{\theta}\right)^n & 0 \le x \le \theta \\ 1-(1-\theta)\left(\frac{1-x}{1-\theta}\right)^n & \theta \le x \le 1 \end{cases} \tag{5}$$

and the density

$$f_X(x) = \begin{cases} n\left(\frac{x}{\theta}\right)^{n-1} & 0 < x \le \theta \\ n\left(\frac{1-x}{1-\theta}\right)^{n-1} & \theta \le x < 1, \end{cases} \tag{6}$$

where $0 \le \theta \le 1$, $n > 0$. The mean and the variance of the random variable X are

$$E[X|\theta,n] = \frac{(n-1)\theta+1}{n+1} \tag{7}$$

and

$$Var(X|\theta,n) = \frac{n-2(n-1)\theta(1-\theta)}{(n+2)(n+1)^2}, \tag{8}$$

respectively. We shall denote this family using the abbreviation $TSP(\theta,n)$. This family of bounded continuous distributions is so flexible and versatile that it is very likely that the FGM family with the TSP marginals may serve as a reliable indicator of the structure of FGM distributions with arbitrary "non-pathological" marginals defined on a compact set. The flexibility of the TSP distribution is similar to that of the beta distribution and both families include U-Shaped $(n < 1, \theta \in (0,1))$, J-Shaped $(\theta \in \{0,1\})$ and unimodal ($n > 1$, $\theta \in (0,1)$ or $\theta \in \{0,1\}$) forms and posses the same limiting distributions (See, Van Dorp and Kotz (2001a)). The uniform distribution $(n = 1)$, power distribution ($\theta = 1$) and triangular distribution $(n = 2)$ all belong to the $TSP(\theta,n)$ family. Similarly to the power distribution the TSP family is not a location-scale family thus yielding a product moment correlation coefficient function of the marginal parameters (unlike the uniform, normal, exponential and lognormal distributions). In

Section 3, the correlation coefficient for the FGM family with TSP marginals is derived as a function of the parameters. We investigate the extremes of this correlation coefficient and its behavior in Section 4. Finally, we shall provide some concluding remarks in Section 5.

2. THE DENSITY

Let $f_{X_1,X_2}(x_1, x_2)$ be the joint density function given by (4) where X_i ($i = 1, 2$) are $TSP(\theta_i, n_i)$ random variables with cdf and pdf given by (5) and (6) respectively. Figures 1 provide some illustrative examples of the joint density $f_{X_1,X_2}(x_1, x_2)$ with identical marginals for a variety of $TSP(\theta_i, n_i)$ scenarios with $\alpha = 1$. Figure 1A displays the bivariate FGM with uniform marginals $(n_i = 1)$, Figure 1B depicts the bivariate FGM with power marginals ($\theta_i = 1, n_i = 2$), Figure 1C displays the bivariate FGM with symmetric triangular marginals ($\theta_i = \frac{1}{2}, n_i = 2$) and Figure 1D presents the bivariate FGM with genuine symmetric TSP marginals ($\theta_i = \frac{1}{2}, n_i = 3$). Figures 2 provide some additional examples of the joint density $f_{X_1,X_2}(x_1, x_2)$ with $\alpha = 1$ and non-identical $TSP(\theta_i, n_i)$, $i = 1, 2,$ marginals. Figure 2A displays the FGM density with TSP marginals with $n_1 = 1$, $\theta_2 = 1, n_2 = 2$. Figure 2B depicts the FGM density with TSP marginals with $n_1 = 1$, $\theta_2 = \frac{1}{2}, n_2 = 2$; Figure 2C presents the FGM density with TSP marginals with $\theta_1 = 0, n_1 = 2$, $\theta_2 = 1, n_2 = 2$. Figure 2D displays the FGM density with symmetric but unequal TSP marginals where $\theta_1 = \frac{1}{2}, n_1 = \frac{1}{2}, \theta_2 = \frac{1}{2}, n_2 = 2$.

It is evident from Figures 1 and 2 that the form of the joint density functions of an FGM distribution with TSP marginals can take great variety of forms. It is therefore of interest to investigate the corresponding correlation structure.

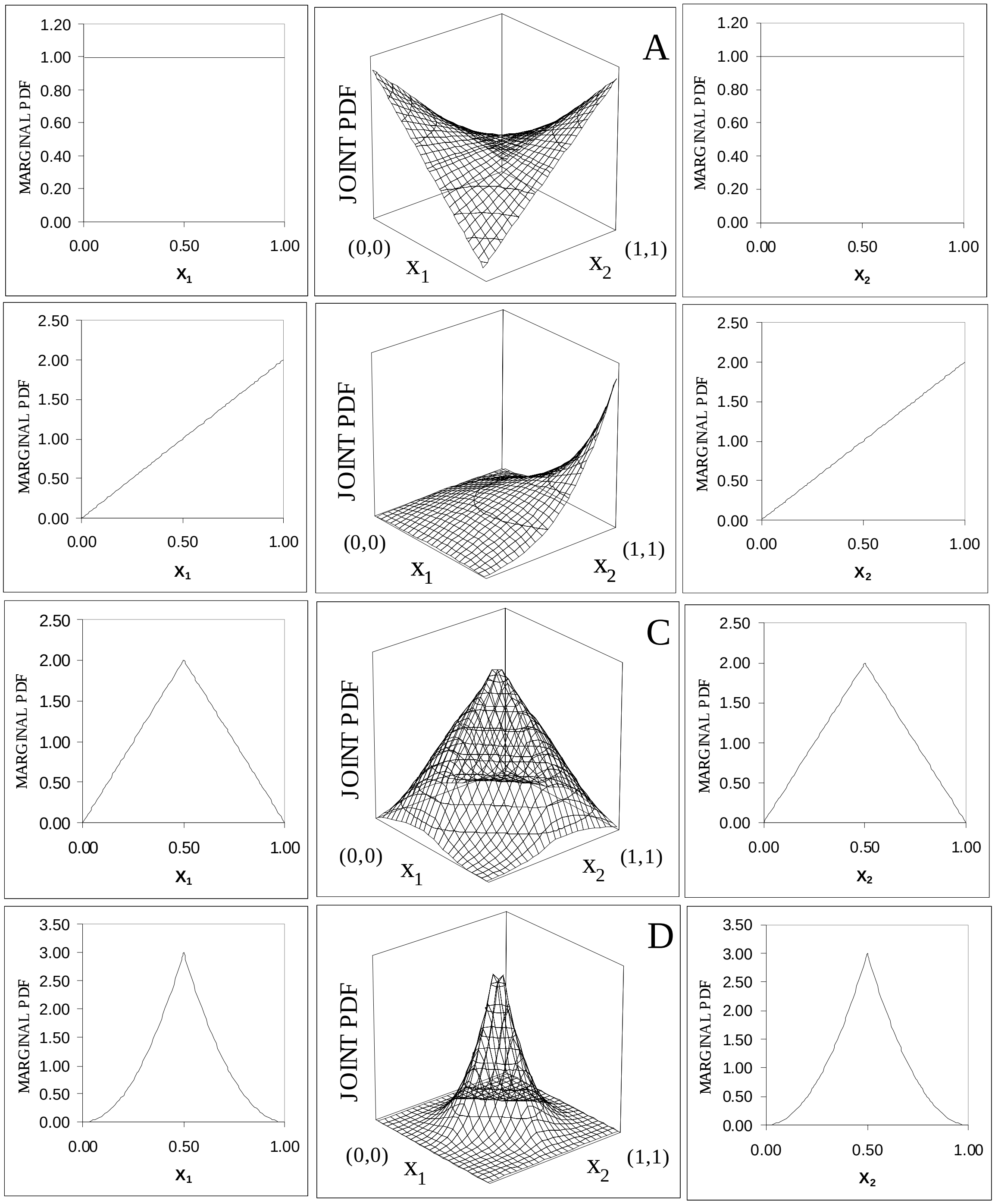

Figure 1. FGM distributions with identical $TSP(\theta_i, n_i)$ marginals, $\alpha = 1$;
A: $n_i = 1$ (uniform); B: $\theta_i = 1, n_i = 2$; C: $\theta_i = 0.5, n_i = 2$; D: $\theta_i = 0.5, n_i = 3, i = 1, 2.$

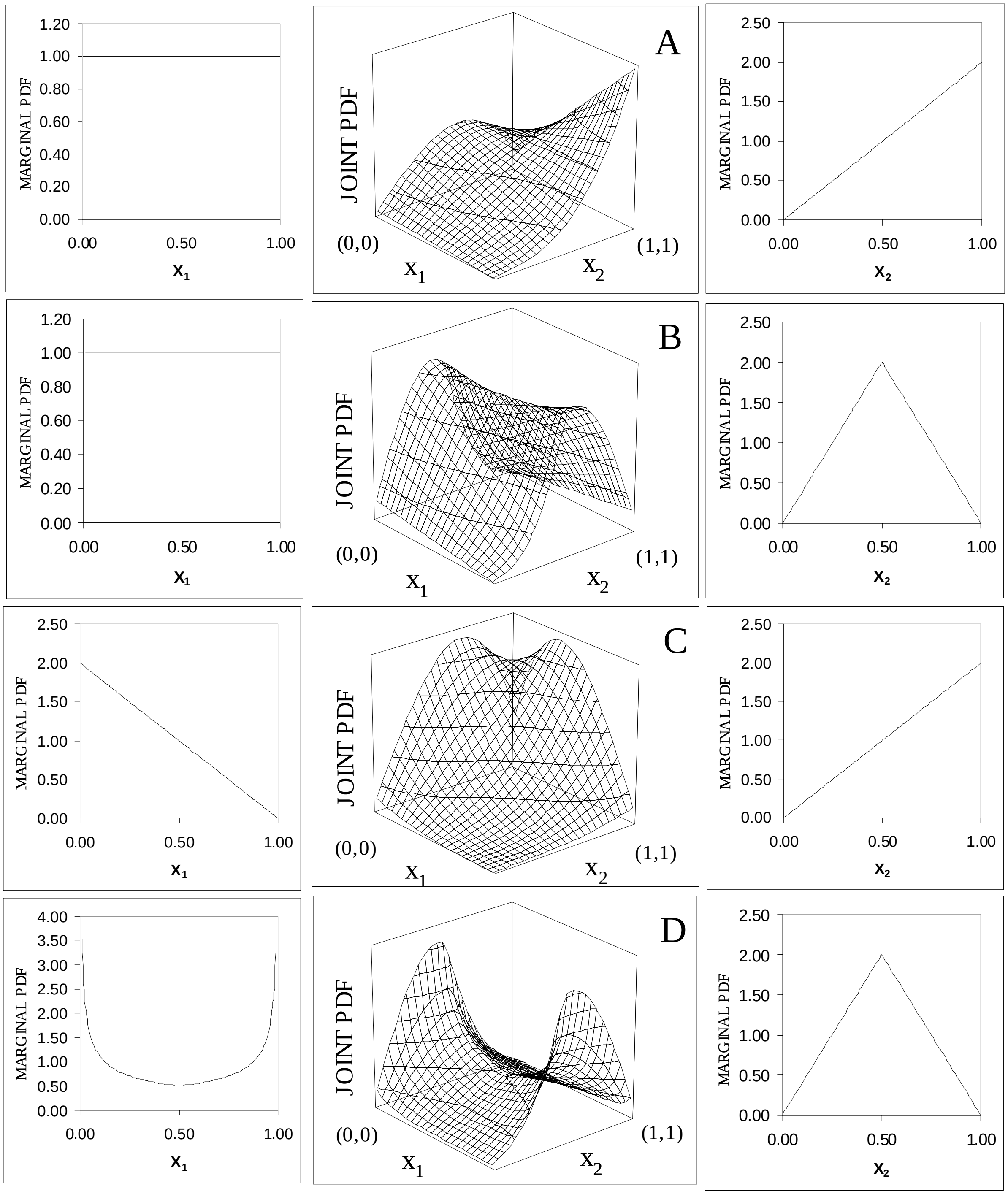

Figure 2. FGM distributions with $TSP(\theta_i, n_i)$ marginals, $\alpha = 1$;
A: $n_1 = 1$, $\theta_2 = 1$, $n_2 = 2$; B: $n_1 = 1$, $\theta_2 = 0.5$, $n_2 = 2$;
C: $\theta_1 = 0$, $n_1 = 2$, $\theta_2 = 1$, $n_2 = 2$; D: $\theta_1 = 0.5$, $n_1 = 0.5$, $\theta_2 = 0.5$, $n_2 = 2$.

3. CORRELATION STRUCTURE

Figures 3A and 3B show the FGM distribution with uniform marginals for the extreme cases of positive dependence ($\alpha = 1$ and $\alpha = 0$). Figure 3B represents the case of independent uniform marginals and the effect of degree of dependence in the joint density may visually be observed by comparing Figures 3A and 3B. The form of the joint density in Figure 3A shows that large (small) values of X_1 tend to be associated with large (small) values of X_2 and thus exhibits positive dependence. Figures 3C and 3D depict the FGM distribution with symmetric triangular marginals for the cases of $\alpha = 1\,(0)$. Figure 3D represents the independent case between the triangular marginals. However, comparing Figures 3C and 3D, it is evident that the degree of dependence in Figure 3C is obscured (due to a non-uniform form of the marginals) even though the dependence parameter α of the FGM distribution has here the same value as in Figure 3A (representing uniform marginals).

The basic (and time honored) measure of dependence between two variables X_1 and X_2 is the covariance

$$Cov(X_1, X_2) = E[(X_1 - E[X_1])(X_2 - E[X_2])]. \tag{9}$$

Among its other properties it *'filters'* the location information contained in the marginal distributions. To make the measure independent of the units with which the variables are expressed, the covariance is divided by the product of the standard deviations leading to the well known product moment (or Pearson's) correlation coefficient:

$$Corr(X_1, X_2) = \frac{Cov(X_1, X_2)}{\sqrt{Var(X_1)Var(X_2)}}. \tag{10}$$

Equation (10) shows that this coefficient also *'filters'* uncertainty information contained in the marginals distribution, thereby separating the assessment of dependence via (10) from the assessment of the location of the marginals (via $E[X_i]$) and the assessment of uncertainty in the marginals (via $Var(X_i)$). The product moment correlation is invariant under arbitrary non-decreasing <u>linear</u> transformations of $X_i,\ i = 1, 2,$ and has been the basic measure of linear dependence for over 100 years. Several other measures have been proposed in the 20-th century to measure positive (or negative) dependence such as Spearman's rank correlation, Kendall's tau, Blomquist's q and Höffding's Δ. (see, e.g., Joag-Dev (1984)). These measures are invariant under all non-decreasing transformations of X_1 and X_2. The product moment correlation is not invariant under arbitrary non-decreasing transformations and thus should be used with some caution as a global measure of positive dependence.

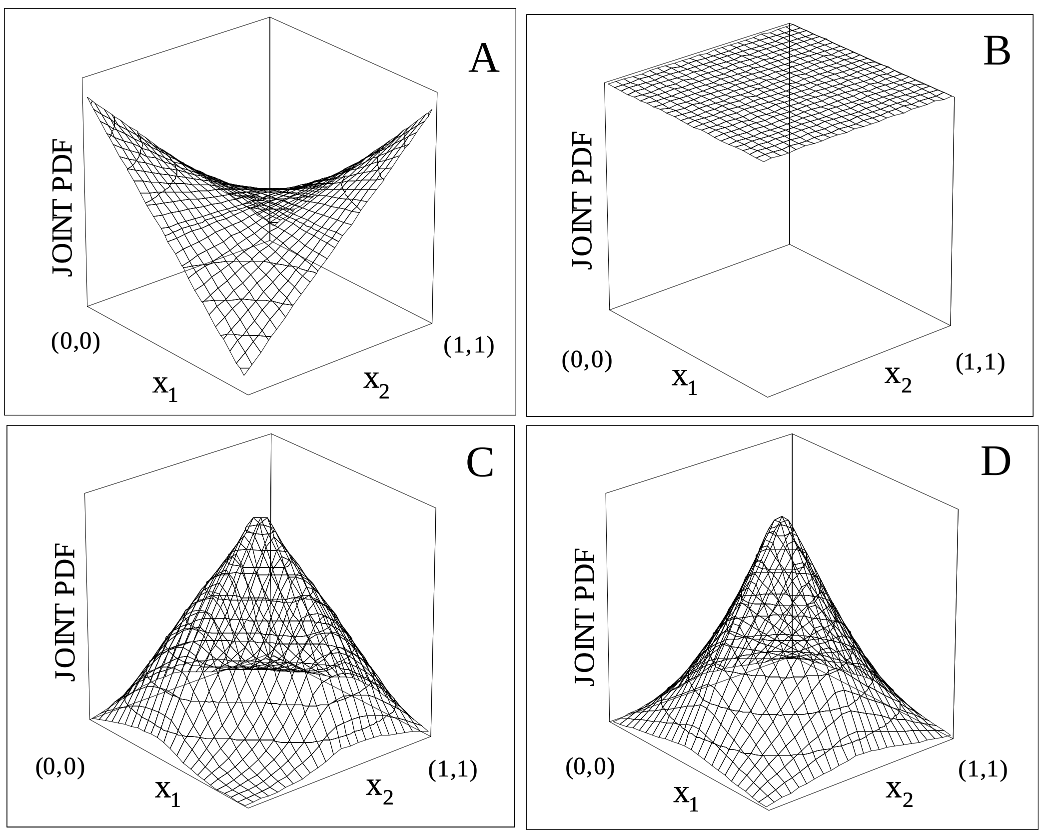

Figure 3. FGM distributions with $TSP(\theta_i, n_i)$ marginals;
A: Dependent uniform marginals ; B: Independent uniform marginals ;
C: Dependent triangular marginals; D: Independent triangular marginals.

We have already pointed out that the values of the correlation coefficient for various bivariate FGM distributions provided in the literature (see, e.g., Schucany et al. (1978)) unequivocally state that the coefficient is $\alpha/4$ for exponential, α/π for normal, 0.281α for Laplace marginals regardless of the values of the parameters of the marginal distributions. These distributions form a location-scale family and thus can be transformed to the standardized form (parameter-free) by a linear transformation resulting in a single value (for a given α) of the correlation coefficient for the whole family regardless of the values of the parameters. This is not the case for the bivariate FGM with TSP marginals where functional dependence on the parameters of the distribution is present. From the identity due to Höffding (1940)

$$Cov(X_1, X_2) = \int_0^1 \int_0^1 \{F_{X_1,X_2}(x_1, x_2) - F_{X_1}(x_1)F_{X_2}(x_2)\}dx_1 dx_2. \tag{11}$$

and utilizing (3) and (10) it follows that for a bivariate FGM distribution with any marginals

$$Cor(X_1, X_2) = \alpha \prod_{i=1}^{2} \frac{h_{X_i}}{\sqrt{Var(X_i)}}, \tag{12}$$

where

$$h_{X_i} = \int_0^1 F_{X_i}(x)\{1 - F_{X_i}(x)\}dx. \tag{13}$$

Substituting the $TSP(\theta_i, n_i)$ cdf (cf. (5)) into (13) we have

$$h_{X_i}(\theta_i, n_i) = \frac{n_i - (n_i - 1)\theta_i(1 - \theta_i)}{(n_i + 1)(2n_i + 1)} \tag{14}$$

and utilizing (8) and (12) we arrive at the correlation coefficient

$$Cor(X_1, X_2|\underline{\theta}, \underline{n}) = \alpha \prod_{i=1}^{2} \sqrt{\frac{(n_i + 2)}{n_i - 2(n_i - 1)\theta_i(1 - \theta_i)}}\ \frac{n_i - (n_i - 1)\theta_i(1 - \theta_i)}{(2n_i + 1)}, \tag{15}$$

where $\underline{\theta} = (\theta_1, \theta_2)$ and $\underline{n} = (n_1, n_2)$.

4. GENERAL BEHAVIOR OF CORRELATION

We shall now investigate the general behavior of (15) as a function of the parameters (θ_i, n_i) ($i = 1, 2$). From the form of (15) it follows that $Corr(X_1, X_2|\underline{\theta}, \underline{n})$ is a separable function in pairs (θ_i, n_i) and it is sufficient to study the general behavior of the function $g(\theta, n)$ given by

$$g(\theta, n) = \frac{(n+2)}{(2n+1)^2} \; \frac{\{n-(n-1)\theta(1-\theta)\}^2}{n-2(n-1)\theta(1-\theta)} \tag{16}$$

$$= \frac{n(n+2)}{(2n+1)^2} \left\{1 + \frac{(n-1)^2\theta^2(1-\theta)^2}{n\{n-2(n-1)\theta(1-\theta)\}}\right\}$$

to assess the effect of the marginals on the product moment correlation. Since $g(\theta, n)$ given by (16) coincides with the correlation of an FGM distribution with identical TSP marginals with parameters $\theta_i = \theta,\ n_i = n$, $i = 1, 2$, and the dependence parameter $\alpha = 1,$ the extremes of $g(\theta, n)$ coincide with extremes of the product moment correlation (cf. (15)).

The first factor $n(n+2)/(2n+1)^2$ in (16) is just the correlation between the power marginals ($\theta = 1$) and determines the general behavior of the product moment correlation coefficient as a function of n. The second term in (16) may be interpreted as a correction factor to account for increased symmetry in the TSP marginals relative to the skewed power marginals. Since the variance is strictly positive and utilizing (8) it follows that this correction factor is greater than 1 for $\theta \in (0,1), n \neq 1$. For both $\theta = 0$ or $\theta = 1$ the correction factor attains its minimum equal to 1 and hence $g(\theta, n)$ attains (for a fixed n) its minimum $n(n+2)/(2n+1)^2$ at either $\theta = 0$ or $\theta = 1$. Writing

$$\frac{(n-1)^2\theta^2(1-\theta)^2}{n\{n-2(n-1)\theta(1-\theta)\}} = \theta(1-\theta)\frac{(n-1)^2}{\frac{n^2}{\theta(1-\theta)} - 2n(n-1)} \tag{17}$$

it immediately follows that the correction factor is a strictly increasing function in the parameter $\varphi = \theta(1-\theta)$. This implies that $g(\theta, n)$ (cf. (16)) attains its maximum for fixed n at $\theta = \frac{1}{2}$.

Taking the partial derivative in (16) with respect to n yields

$$\frac{\partial}{\partial n} g(\varphi, n) = (n-1)\; \frac{(n+3\varphi)(6\varphi-2)}{(2n+1)^3} \frac{\{n-(n-1)\varphi\}}{\{n-2(n-1)\varphi\}^2}, \tag{18}$$

where as above $\varphi = \theta(1-\theta)$. Since $\theta \in [0,1]$, $\varphi \in [0, \frac{1}{4}]$. Hence, noting that $h_{X_i}(\theta_i, n_i)$ given by (14) is strictly positive it follows from (18) that for fixed φ, $g(\varphi, n)$ is a strictly increasing function of n for $0 < n \leq 1$ and a strictly decreasing one for $n \geq 1$. Hence for any fixed θ the function $g(\theta, n)$ attains its maximum at $n = 1$. Since $g(\theta, 1) = 1/3$ we can conclude that the product moment correlation of the FGM with TSP marginals attains its global maximum $\alpha/3$ when both TSP marginals are uniform. This agrees with the general result in Schucany et al. (1978), already mentioned above, that the maximum correlation of an FGM distribution is attained for uniform marginals. Also it follows from (18) that for a fixed φ (and thus a fixed θ), the minimal correlation is attained either when $n \downarrow 0$ or $n \to \infty$. Utilizing (16) we derive

$$g(\theta, 0) = \underset{n \downarrow 0}{Lim}\, g(\theta, n) = \theta(1-\theta) \tag{19}$$

$$g(\theta, \infty) = \underset{n \to \infty}{Lim}\, g(\theta, n) = \frac{1}{4}\left\{1 + \frac{\theta^2(1-\theta)^2}{1-2\theta(1-\theta)}\right\}. \tag{20}$$

Figures 4 summarize the extremes of the function $g(\theta, n)$ as a function of the parameter θ (n) keeping n (θ) fixed. Figure 4A focusses on the extremes of $g(\theta, n)$ as a function of θ for values of $n \le 1$ and displays $g(\theta, 1) = 1/3$ (the global maximum) and $g(\theta, 0)$ (cf. (19)) (the minimal value of $g(\theta, n)$ for fixed θ with $n \le 1$). Analogously, Figure 4B focusses on the extremes of $g(\theta, n)$ as a function of θ for $n \ge 1$, displays $g(\theta, 1) = 1/3$ and $g(\theta, \infty)$ (cf. (20)) (the minimal value of $g(\theta, n)$ for fixed θ with $n \ge 1$). Figure 4C presents the extremes of $g(\theta, n)$ as function of n keeping θ fixed and displays $g(\frac{1}{2}, n)$ (the maximum for fixed n) and $g(0, n) = g(1, n)$ (the minimal value for fixed n). Figure 5 provides a three dimensional graph of $g(\theta, n)$. Figures 4 A, B and C may be interpreted as two-dimensional $projections$ of Figure 5 in various directions.

From (16) and Figures 4 it follows that the correlation is bounded from above by the product moment correlation $\alpha/3$ of the FGM distribution with uniform marginals and from below by $\alpha n(n+2)/(2n+1)^2$ $-$ the correlation of the FGM distribution with power marginals $(\theta = 1)$, or its reflection ($\theta = 0$). From Figures 4A and 4B it follows that for fixed n_1 and n_2 the correlation attains it maximal value when both TSP marginals are symmetric ($\theta = \frac{1}{2}$) and its minimal value when both TSP marginals attain the maximal skewness ($\theta \in \{0, 1\}$). Figure 4C shows that the maximum discrepancy between the correlation with identical power marginals with $n_i = n$ and identical TSP marginals with $n_i = n$ is equal to

$$d(n) = g(\frac{1}{2}, n) - g(0, n) = \frac{1}{8}\frac{n+2}{n+1}\left\{\frac{n-1}{2n+1}\right\}^2. \tag{21}$$

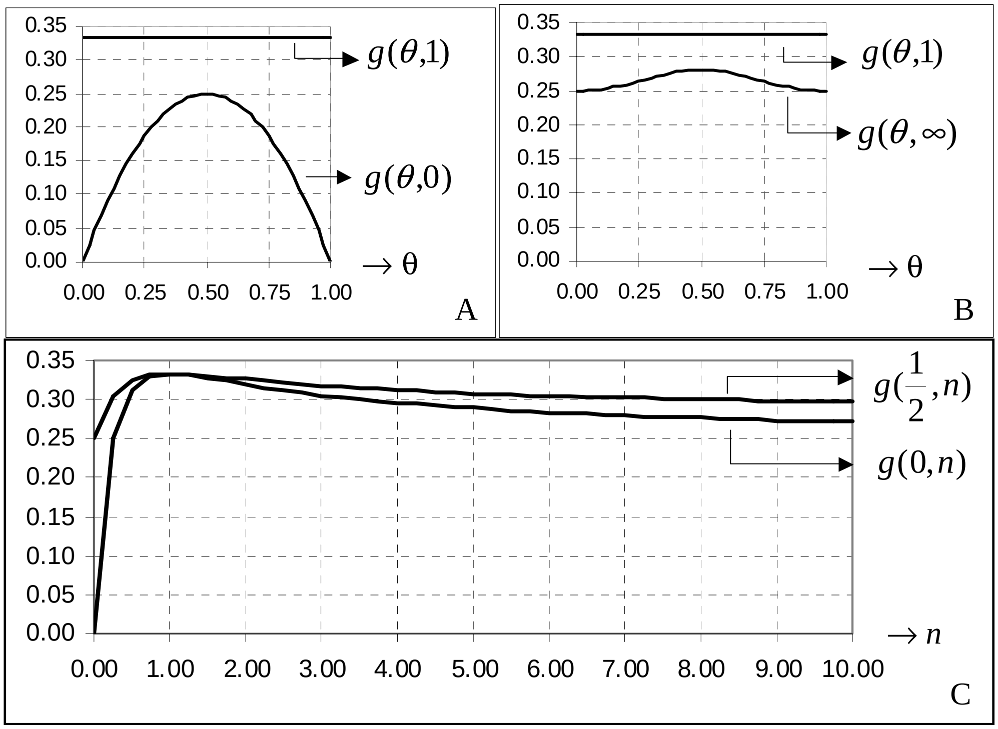

Figure 4. The extremes of the function *g(θ, n);* A: as a function of $\theta\ (0 < n \le 1)$; B: as a function of $\theta\ (n \ge 1)$; C: as a function of $n\ (\theta \in [0,1])$.

The function $d(n)$ describes, for fixed n, the maximum discrepancy in the correlation due to the skewness parameter θ; it is a strictly decreasing (increasing) function of n on $[0,1]$ ($[1,\infty)$). For $n \to \infty$, $d(n) \to \frac{1}{32}$ $(= d(\infty))$and the largest discrepancy is attained at $n = 0$. Apparently (see Figure 4C) the difference between the correlations of FGM distributions with identical TSP marginals and identical power marginals is the largest for small values of n. However, for

$$n^* = \frac{-17 + \sqrt{513}}{16} \approx 0.35, \tag{22}$$

$d(n^*) = d(\infty) = \frac{1}{32}$. Hence, for all values of $n > n^*$ (which are plausible in applications) the discrepancy $d(n) < \frac{1}{32} = 0.03125$. From (16) it follows that $g(0, n^*) \approx 0.28541$. Thus we have reached the important conclusion that for $n > n^* \; g(\theta, n)$ varies from $g(0, n^*)$ to $g(\theta, 1)$ (the uniform case) at most $\approx (\frac{1}{3} - 0.28541) \approx 0.04792$.

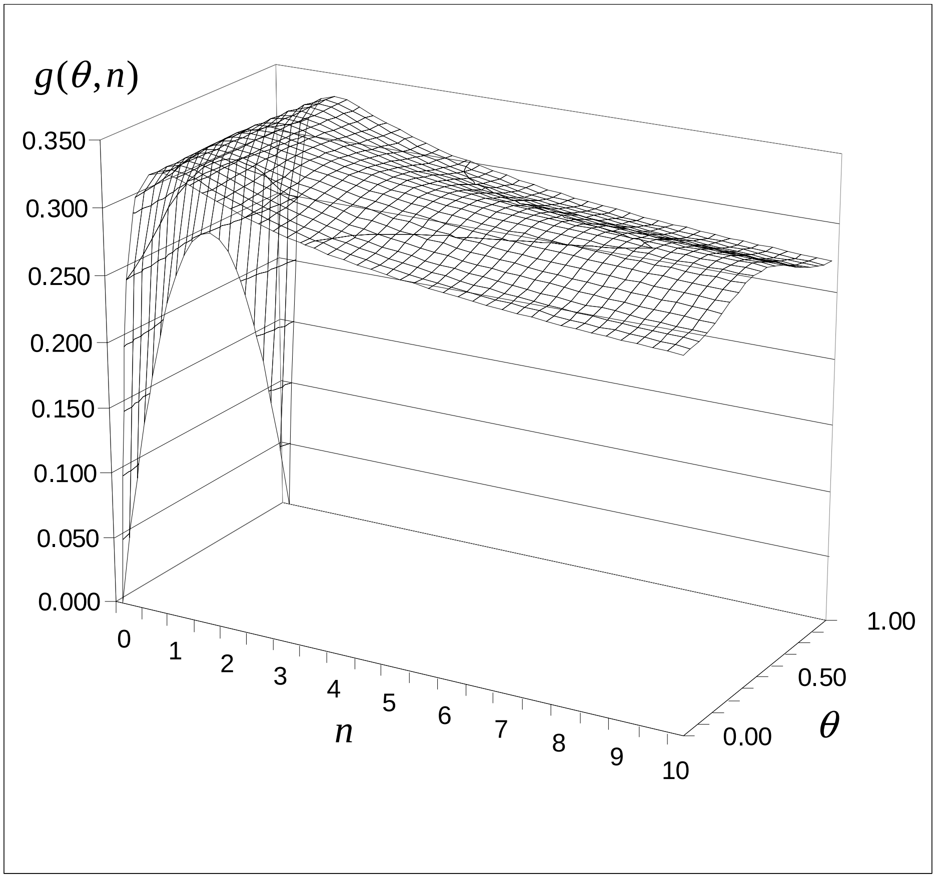

Figure 5. General behavior of function *g(θ, n)* given by *(16)* as a function of *θ* and *n.*

Summarizing the function $g(\theta, n)$ for values of $n > n^*$ varies in the narrow range of ≈ 0.04792 from the correlation coefficient of an FGM distribution with uniform marginals. Moreover, for fixed $n > n^*$, the correlation coefficient of an FGM with identical TSP marginals is strictly larger than the correlation coefficient of an FGM with identical power marginals, but differs from it by no more than $\frac{1}{32} = 0.03125$ for all values $\theta \in [0, 1]$. Hence, the effect of the parameters θ and n on the value of the product moment correlation is almost negligible for practical purposes. The general behavior of the product moment correlation with TSP marginals closely follows the behavior of the FGM with power marginals, i.e. it increases for $n \in (0, 1]$ and decreases for $n \in [1, \infty)$. As the values of the function $g(\theta, n)$ are remarkably stable for $n \geq n^*$ it may be concluded that the product moment correlation is especially stable for values of $n_i \geq n^*$. Also from the above analysis the product moment correlation for the FGM distribution with beta marginals (which cannot be derived in closed form) quite likely closely mirrors the behavior of the FGM distribution with TSP marginals exhibiting similar stability.

5. CONCLUSIONS

This paper studied a new class of bivariate FGM distribution with TSP marginals. The TSP marginals have bounded support and include the uniform, power and triangular distributions. The latter distribution is especially popular amongst engineers, yet modeling dependence amongst triangular distributions has not been investigated so far. Independence assumption amongst the marginals which is often encountered in the literature is dubious and largely motivated by convenience. The new class of FGM distributions with TSP marginals encourages modeling of mild dependence in practical applications which occur in e.g. the fields of medicine and engineering. Using a linear transformation, a TSP distribution with support $[0, 1]$ may be transformed to TSP distributions with support $[a, b]$. The parameters of the transformed TSP distributions involve a lower bound a, the mode $m = (b - a)\theta + a$ and an upper bound b, (as is the case for the triangular distribution) as well as the shape parameter n. The correlation structure of the FGM distributions with TSP marginals is not affected by these linear transformations of marginals.

The product moment correlation coefficient for the FGM family with TSP marginals depends, however, on the values of the parameters of the marginal distributions. Interestingly enough, these values exhibit substantial stability, hovering between roughly $\alpha/4$ and the maximum possible correlation $\alpha/3$, for all plausible values of the parameters θ_i and n_i, $i = 1, 2$. Only in the limiting cases we have divergent values. The product moment correlation is strictly speaking a measure of linear dependence and is not invariant under general non-decreasing transformations. However, the discovered stability of the product moment correlation under a wide variety of marginals of diverse forms in the FGM distributions suggests that for bivariate distributions with bounded support, this coefficient also is a suitable measure of positive (or negative) dependence. An extension of these results to generalized FGM distributions (with larger correlation coefficients) could be of some interest and utility.

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