A General Bayes Weibull Inference Model for Accelerated Life Testing

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ABSTRACT: This article presents the development of a general Bayes inference model for accelerated life testing. The failure times at a constant stress level are assumed to belong to a Weibull distribution, but the specification of strict adherence to a parametric time-transformation function is not required. Rather, prior information is used to indirectly define a multivariate prior distribution for the scale parameters at the various stress levels and the common shape parameter. Using the approach, Bayes point estimates as well as probability statements for use-stress (and accelerated) life parameters may be inferred from a host of testing scenarios. The inference procedure accommodates both the interval data sampling strategy and type I censored sampling strategy for the collection of ALT test data. The inference procedure uses the well-known Markov Chain Monte Carlo (MCMC) methods to derive posterior approximations. The approach is illustrated with an example.

1 INTRODUCTION

In the case of highly reliable items, e.g. Very Large Scale Integrated (VLSI) electronic devices, computer equipment, missiles, etc., mean times to failure (MTTF) exceeding a year is not uncommon. The use of these items, however, may still require reliability demonstration or verification testing, especially when these items are used for military or high risk public applications. With such MTTF's, it is often too time consuming and too costly to test these items in their use (or nominal) environment, as the length of time to generate a reasonable number of failures is often intolerable. If such is the case, it has become a standard procedure (see MIL-STD-781C) to test these items under more severe environments than experienced in actual use. Such tests are often referred to as Accelerated Life Tests (ALT's). Mann and Singpurwalla (1983) note that because of advancement in technology and increased reliability, ALT's are performed more frequently than ordinary life tests. There are two main problems associated with ALT's as: (1) optimal design of the ALT, and (2) statistical inference from ALT failure data. The focus of this paper is on the statistical inference problem, i.e. on how to make inference about the reliability in the use environment by obtaining information in the accelerated environments.

Typically inference methods have been developed assuming that: (1) the life time distribution in a constant stress environment belongs to a common family of distributions, and (2) the scale parameter of such a distribution is related to the stress environment via a parametric function known as a time transformation function (TTF) (see for example Mann et al (1974)). In addition, most of the inference methods are based on the use of maximum likelihood estimation which may require large sample sizes for meaningful statistical ALT inference (see for example Nelson (1980)). In this paper, only the first assump-tion will be adhered to. Specifically, inference will be developed using the Weibull failure time model. The inference method is Bayesian in nature and will rely on the use of engineering judgment to specify prior distributions for the Weibull model parameters. While there is a host of literature in this area, the only Bayesian inference procedure developed for the Weibull model that we know of is presented in Mazzuchi et al (1997) for constant stress ALTs in conjunction with the parametric TTF. The inference procedure herein will be developed for a wide range of ALT scenarios with no TTF assumption.

In Section 2 the general likelihood model is developed. In Section 3 the prior distribution for the shape parameter and scale parameters of the Weibull failure time model is outlined. The posterior inference is briefly discussed in Section 4. The approach is illustrated by an example in Section 5.

2 A GENERAL LIKELIHOOD MODEL

2.1 *Motivation*

A first step in any statistical inference procedure, whether classical or Bayesian involves developing the likelihood. The flexibility of the likelihood formulation drives the flexibility of the statistical inference procedure in terms of its applicability to different ALT scenarios. In this section, a likelihood model is developed that allows for a comprehensive representation of most ALT inference scenarios currently available to ALT practitioners, specifically, regular life testing, fixed-stress testing, and progressive step-stress testing. In addition, the likelihood model allows for profile step-stress testing and different ALT patterns for each test item as illustrated in Figure 1. Having such a flexible formulation of a likelihood model allows for the comparison of different ALT designs within a common modeling framework. In addition, allowing for such a flexibility will increase the model's ability to represent ALT designs used by testing practitioners.

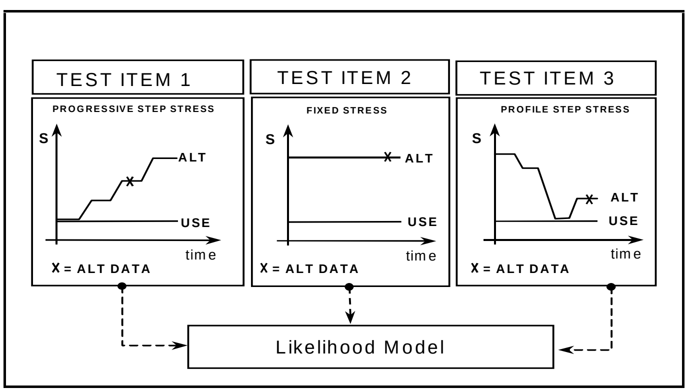

Figure 1. A Separate ALT Design for Each Test Item

2.2 *The Failure Rate over the Course of an ALT*

In developing a likelihood model, consider the following step-stress ALT setup. The ALT will consist of testing over a predetermined and fixed maximum number of test environments. An environment is defined by a combination of stress levels from the set of stress variables, e.g. temperature, vibration and voltage. Let K environments $E_1, \ldots, E_K$ be defined candidate test environments. Let E_ϵ denote the use environment, where $\epsilon \in \{1,\ldots,K\}$. The index e will be used to indicate a particular environment. Suppose that each of N test items will be subjected to a step-stress ALT with possibly a different step pattern per test item. The index j will be used to indicate a particular test item. The total length of each ALT may vary per test item, but each ALT will be subdivided into m steps. The index i will be used to indicate a particular step-interval within an ALT. Thus, for each test item j, m steps are defined by

$$0 \equiv t_{0,j} < t_{1,j} < \ldots < t_{m-1,j} < t_{m,j}$$

where the i-th step is defined as $[t_{i-1,j}, t_{i,j})$ and the ALT is terminated at time $t_{m,j}$ for test item j. A design matrix $A = \{a_{i,j}\}$ specifies the indices of the environments for each test item j in each step i. Thus during the i-th step, test item j is subjected to environment $E_{a_{i,j}}$ where $a_{i,j} = 1, \ldots, K$. Note that this flexible formulation includes both regular life testing ($a_{i,j} \equiv \epsilon$ for all i) and fixed-stress ALT ($a_{i,j} \equiv z$ for all i and some particular environment $E_z, z \neq \epsilon$).

The approach to deriving the likelihood will be general and center around the failure rate, $\lambda_e(t)$, in a constant stress environment E_e. The failure rate for test item j over the course of the step-stress ALT, denoted by $h_j(t)$, is different from the failure rates in the constant stress environments, as the environments, and thus the failure behavior, vary over the course of the ALT stage. A generic expression will be derived for the failure rate $h_j(t)$ of test item j over the course of an ALT stage, conditioned on knowing the failure rate functions $\lambda_e(t)$ in the candidate test environments E_e, $e = 1, \ldots, K$.

The cumulative failure rate that test item j has accumulated up to $t_{i,j}$ in an ALT is given by

$$H_j(t_{i,j}) = \int_0^{t_{i,j}} h_j(u)du. \quad (1)$$

In a constant stress environment E_e, the cumulative failure rate would be given by

$$\Lambda_e(t_{i,j}) = \int_0^{t_{i,j}} \lambda_e(u)du. \quad (2)$$

Note that the operating environment in the ALT after $t_{i,j}$ equals $E_{a_{i+1,j}}$. It will be assumed that the change in environments at $t_{i,j}$ is assumed to be instantaneous. In addition, we assume that no additional failures are induced by the instantaneous change of environments $t_{i,j}$ through a shock effect.

Using (2), the cumulative hazard rate $H_j(t_{i,j})$ may be expressed using $\Lambda_{a_{i+1,j}}(t)$ for some value of t. Denoting this value t by $\tau_{i+1,j}$, and solving for $\tau_{i+1,j}$, yields

$$\tau_{i+1,j} \;=\; \Lambda^{-1}_{a_{i+1,j}}(H_j(t_{i,j})). \tag{3}$$

The time $\tau_{i+1,j}$ may be interpreted as the amount of time that would have elapsed to accumulate $H_j(t_{i,j})$ by testing in environment $E_{a_{i+1,j}}$ alone, starting at time 0 (see for example Figure 2).

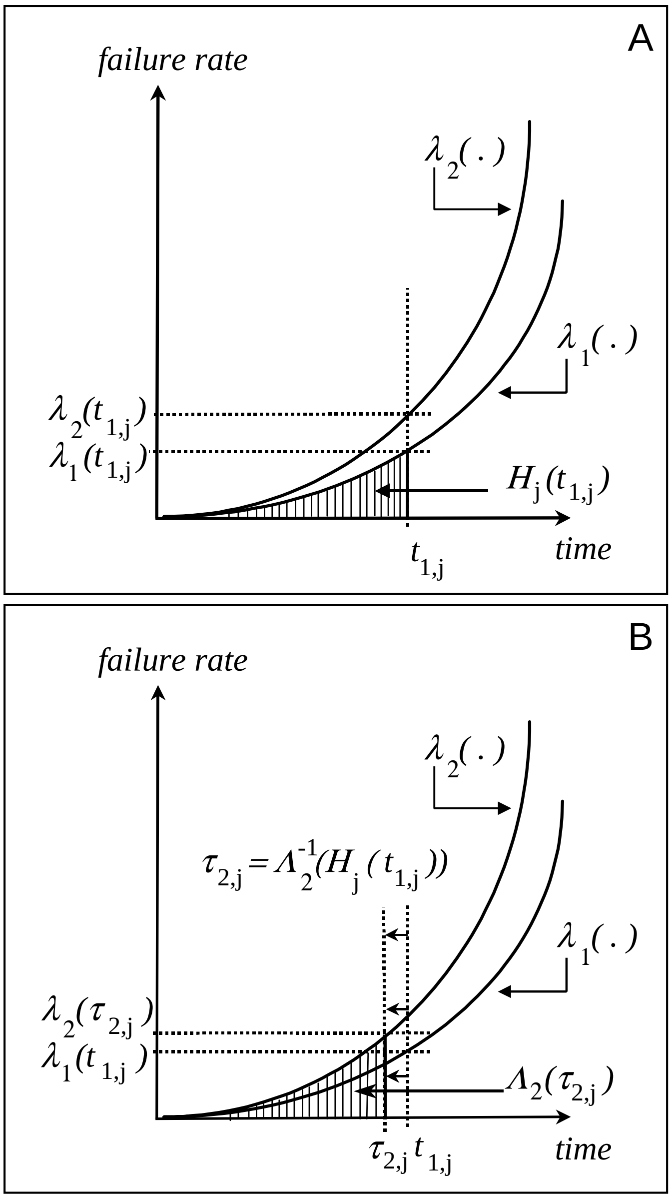

Figure 2. Failure Rate Construction Using Instantaneous Environment Changes. A: Failure Rate up to $t_{1,j}$; B: Change Failure Rate to $\lambda(\tau_{2,j})$ at $t_{1,j}$ such that $\Lambda_2(\tau_{2,j}) = H_j(t_{2,j})$.

Next, the failure rate function over the course of the ALT stage may be derived as

$$h_j(t) = \lambda_{a_{i+1,j}}(t - t_{i,j} + \tau_{i+1,j}), \tag{4}$$

for $t_{i,j} \le t < t_{i+1,j}$, $i = 0, \ldots, m-1$, where $\tau_{i+1,j}$ is given by (3) and $H_j(t_{0,j})$ is the initial cumulative failure rate of test item j prior to the ALT stage. In case $H_j(t_{0,j})$ is a new test item, $H_j(t_{0,j}) \equiv 0$. However, the case where a test item has a history of operating hours in known environments may be easily accommodated. The jump in the failure rate at time $t_{i,j}$ follows as

$$\Delta h_j(t_{i,j}) = h_j(t^+_{i,j}) - h_j(t^-_{i,j}), \tag{5}$$

where $h_j(t^-_{i,j}) = \lim_{t \uparrow t_{i,j}} h_j(t)$, $h_j(t^+_{i,j}) = \lim_{t \downarrow t_{i,j}} h_j(t)$. It may be derived that the jump $\Delta h_j(t_{i,j})$ equals

$$\Delta h_j(t_{i,j}) = \\ \lambda_{a_{i+1,j}}(\tau_{i+1,j}) - \lambda_{a_i}(t_{i+1,j} - t_{i,j} + \tau_{i,j}) \tag{6}$$

for $i = 1, \ldots, m-1$. Figure 3 presents an example of the above construct for a profile ALT sequentially stepping through the enviroments E_1, E_3, E_5, E_4 and E_2 where a Weibull failure time distribution is assumed for each constant stress. Following the approach above, the current failure rate of a test item only depends on the current accumulated cumulative failure rate and the current stress. It can be shown that the *intrinsic time* failure rate construction approach above is equivalent to the assumption of the Linear Cumulated Damage (LCD) model (see for example Nelson (1980)).

The assumption that no additional failures are induced by the instantaneous change of environments between steps is an assumption which may be challenged, as an instantaneous change of environments may induce a shock effect, causing item failure. The above procedure, however, can be easily extended to the case of gradual environmental changes (see for example Van Dorp (1998)).

2.3 *The Likelihood Given ALT Test Data*

Using the formulation of the failure rate function over the course of an ALT-stage, the likelihood given ALT Test data may be derived for both interval and Type I censored data.

2.3.1 Interval Data

Suppose failures can only be monitored at the end of a step interval $i+1$, i.e. $[t_{i,j}, t_{i+1,j})$. The interval in which item j fails will be denoted by q_j. The probability of test item j surviving time $t_{i+1,j}$ given that it has survived up to time $t_{i,j}$ follows as

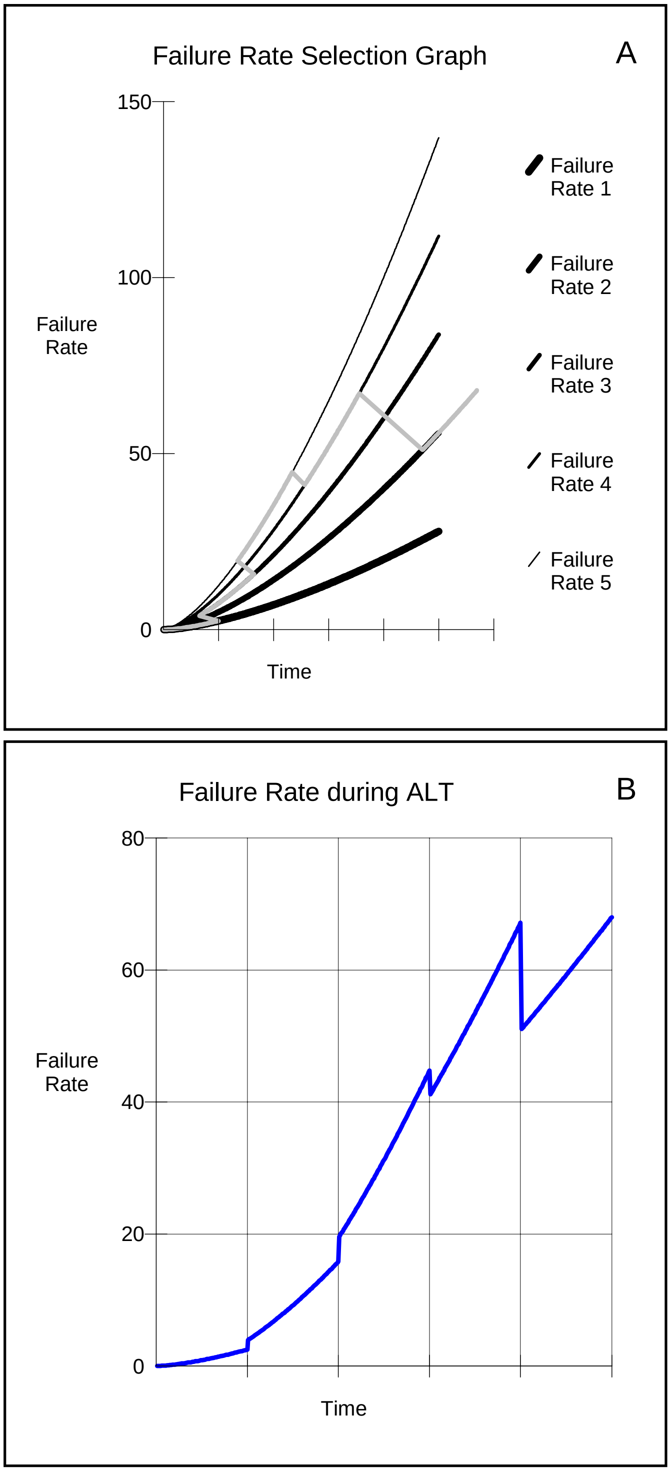

Figure 3. A: Selection of Failure Rate Sections from Weibull Falure Rate Functions B: Failure Rate During a Profile ALT by concatening the failure rate sections selected in A.

$$
\begin{aligned}
&Pr\{T_j \geq t_{i+1,j} | \, T_j \geq t_{i,j}\} \\
&= exp\Big\{ -\int_{t_{i,j}}^{t_{i+1,j}} h_j(u)du \Big\} \\
&= exp\Big\{ -\int_{\tau_{i+1,j}}^{t_{i+1,j}-t_{i,j}+\tau_{i+1,j}} \lambda_{a_{i+1,j}}(v\,)\,dv \Big\},
\end{aligned} \tag{7}
$$

where $\tau_{i+1,j}$ is given by (3). The probability of test item j failing before time $t_{i+1,j}$ given that it has survived up to time $t_{i,j}$ equals

$$
\begin{aligned}
&Pr\{T_j < t_{i+1,j} | \, T_j \geq t_{i,j}\} \\
&= 1 - \; exp\Big\{ -\int_{\tau_{i+1,j}}^{t_{i+1,j}-t_{i,j}+\tau_{i+1,j}} \lambda_{a_{i+1,j}}(v\,)\,dv \Big\}
\end{aligned} \tag{8}
$$

The probability of test item j failing in interval q_j equals

$$
\begin{aligned}
&\prod_{i=1}^{q_j-1} \Big\{ Pr\{T_j \geq t_{i,j} | \, T_j \geq t_{i-1,j}\} \\
&\times Pr\{T_j < t_{q_j,j} | \, T_j \geq t_{q_j-1,j}\} \Big\},
\end{aligned} \tag{9}
$$

with the convention that $q_j = m+1$ if the test item is censored at $t_{m,j}$ and $t_{m+1,j} \equiv \infty$. Substituting (7) and (8) in (9) yields

$$
\begin{aligned}
&p(q_j) \; \times \\
&\prod_{i=1}^{q_j-1} exp\Big\{ -\int_{\tau_{i+1,j}}^{t_{i+1,j}-t_{i,j}+\tau_{i+1,j}} \lambda_{a_{i+1,j}}(v\,)\,dv \Big\}
\end{aligned} \tag{10}
$$

where $p(q_j)$ equals

$$
1 - \; exp\Big\{ -\int_{\tau_{q_j,j}}^{t_{q_j,j}-t_{q_j-1,j}+\tau_{q_j,j}} \lambda_{a_{q_j,j}}(v\,)\,dv \Big\} \tag{11}
$$

for $q_j < m+1$ and is defined as 1 for $q_j \; = \; m+1$

With (10) and assuming conditional independence between the failure times of the test items conditioned on knowing $\underline{\lambda}(\,\cdot\,) = (\lambda_1(\,\cdot\,), \ldots, \lambda_K(\,\cdot\,))$, it follows that the likelihood given interval data $(N, \underline{q}) \; = \; (q_1, \, ..., \, q_N)$, equals

$$
\begin{aligned}
&\mathcal{L}\{\underline{\lambda}(\,\cdot\,);(N, \; \underline{q}\;)\} = \prod_{j=1}^{N} p(q_j) \times \\
&\prod_{i=1}^{q_j-1} exp\Big\{ -\int_{\tau_{i+1,j}}^{t_{i+1,j}-t_{i,j}+\tau_{i+1,j}} \lambda_{a_{i+1,j}}(v\,)dv \Big\}
\end{aligned} \tag{12}
$$

where N is the number of test items in the ALT. Though not specifically developed here, the previous equations may also be adjusted for the case where items begin the ALT with accumulated damage as in the case of retesting of items. This is considered in Van Dorp (1998).

To be able to perform inference with respect to the failure rates $\lambda_e(t)$ in each environment, it is convenient to reorder the product in (12) as a product over the environment index e instead of over the step interval index i. Given $\tau_{i,j}$ $i = 1, \ldots, m-1$ via (3), such a reordering is possible. To accomplish such a formulation, let

$n_{e,j}$ = *the number of times that test item j visits environment E_e during an ALT stage,*
$v^k_{e,j}$ = *interval index for which item j visits E_e for the k-th time.*

With the above notation (12), may be rewritten as

$$\mathcal{L}\{\underline{\lambda}(\cdot);(N,\underline{q})\} = \prod_{e=1}^{K}\prod_{j=1}^{N}\prod_{k=1}^{n_{e,j}} f(e,j,k\,|\,\underline{\lambda}(\cdot)\,,\,\underline{q}\,) \tag{13}$$

where

$$f(e,j,k|\underline{\lambda}(\cdot),\underline{q}) = \begin{cases} f_1(e,j,v^k_{e,j}|\underline{\lambda}(\cdot)) & v^k_{e,j} < q_j \\ f_2(e,j,v^k_{e,j}|\underline{\lambda}(\cdot)) & v^k_{e,j} = q_j \end{cases} \tag{14}$$

and

$$f_1(e,j,v_e|\underline{\lambda}(\cdot)) = exp\Big\{-\int_{\tau_{v_e,j}}^{t_{v_e,j}-t_{v_e-1,j}+\tau_{v_e,j}} \lambda_e(v)\,dv\Big\}, \tag{15}$$

and $f_2(e,j,v_e|\underline{\lambda}(\cdot)) = 1 - f_1(e,j,v_e|\underline{\lambda}(\cdot))$ where $\tau_{i,j}$ is given by (3). Note that, f_1 (f_2) is the conditional probability of surviving the step (failing in the step) interval for which test item j visits environment E_e for the k-th time, conditioned on having survived up to the beginning of that step. When assuming a common family of life time distributions within a constant stress environment, i.e. specifying a functional form for $\underline{\lambda}(\cdot)$, the likelihood may be further derived using (13) - (15). Note, that in principle different failure models for different enviroments may be specified.

The interval data sampling strategy has the disadvantage that failure information is lost by only monitoring at the end of each step interval. In the type I censored sampling strategy, test items are continuously monitored over the course of the ALT.

2.3.2 *Type I Censored Data*

Suppose failures can be monitored continuously over he course of an ALT stage. In that case, the failure time r_j of test item j is known exactly if the test item fails in $[0, t_{m,j})$. It will be assumed that once an item has failed, it will be removed from testing in the same ALT Stage. Knowing the failure times r_j, the step intervals q_j in which the items failed may be easily derived. Using an analogous approach as in Section 2.3.1, the likelihood given the data $(N,\ \underline{r},\ \underline{q}$ $)$, where $\underline{r} = (r_1, \ldots, r_N)$, and $\underline{q} = (q_1, ..., q_N)$, follows as specified in the expressions (17)-(20),

$$\mathcal{L}\{\underline{\lambda}(\cdot);(N,\underline{r},\underline{q}\,)\} = \prod_{e=1}^{K}\prod_{j=1}^{N}\prod_{k=1}^{n_{e,j}} g(e,j,k\,|\,\underline{\lambda}(\cdot)\,,\underline{r},\,\underline{q}\,), \tag{17}$$

$$g(e,j,k\,|\,\underline{\lambda}(\cdot),\underline{r},\underline{s},\,\underline{q}\,)= \begin{cases} g_1(e,j,v^k_{e,j}|\,\underline{\lambda}(\cdot)\,,\underline{r}) & v^k_{e,j} < q_j \\ g_2(e,j,v^k_{e,j}|\,\underline{\lambda}(\cdot),\underline{r}\,) & v^k_{e,j} = q_j, \end{cases} \tag{18}$$

$$g_1(e,j,v_e\,|\,\underline{\lambda}(\cdot)\,,\underline{r}) = exp\Big\{-\int_{\tau_{v_e,j}}^{t_{v_e,j}-t_{v_e-1,j}+\tau_{v_e,j}} \lambda_e(v)\,dv\Big\}, \tag{19}$$

$$g_2(e,j,v_e|\,\underline{\lambda}(\cdot),\underline{r}\,) = exp\Big\{-\int_{\tau_{v,j}}^{r_j\,-t_{v_e-1,j}\,+\tau_{v_e,j}} \lambda_e(v)\,dv\Big\}h_j(r_j). \tag{20}$$

Note that, g_1 is the conditional probability of surviving the step interval for which test item j visits environment E_e for the k-th time conditioned on having survived up to the beginning of that step. In addition, note that g_2 is the conditional density at the time of failure in case the test item fails within the step interval for which test item j visits environment E_e for the k-th time conditioned on having survived up to the beginning of that step.

When assuming a common family of life time distributions within constant environments, i.e., specifying a functional form for $\underline{\lambda}(\cdot)$, the likelihood may be further derived using (17) - (20). Such expressions can for example be derived for the Weibull life distribution using $\lambda_e(t) = \lambda_e \beta\, t^{\beta-1}$.

3 PRIOR DISTRIBUTION

Given the ordering of the severity of the testing environments, it is natural to assume that

$$0 \equiv \lambda_0 < \lambda_1 < \ldots < \lambda_K < \lambda_{K+1} \equiv \infty, \tag{21}$$

and, defining

$$u_e = e^{-c\lambda_e} \tag{22}$$

for some constant c, it follows that

$$0 \equiv u_{K+1} < u_K < \ldots < u_1 < u_0 \equiv 1. \tag{23}$$

The parameter c is chosen to insure numerical stability of the results (see Van Dorp and Mazzuchi (2003)). Rather than defining a prior distribution for $\underline{\lambda}$ exhibiting property (21), one may equivalently define a prior for $\underline{u} = (u_1, \ldots, u_K)$ exhibiting property (23). Concentrating on $\underline{u} = (u_1, \ldots, u_K)$, a prior distribution which is mathematically tractable, is defined over the region specified in (23), and imposes no other unnecessary restrictions on the u_e, is the multivariate Ordered Dirichlet distribution,

$$\Pi\{\underline{u} \mid \eta, \underline{\nu}\} = \frac{\prod_{e=1}^{K+1} (u_{e-1} - u_e)^{\eta \cdot \nu_e - 1}}{\mathbb{D}(\eta, \underline{\nu})}, \tag{24}$$

where, $\eta > 0,\ \nu_e > 0,\ e = 1, \ldots, K+1,$ and

$$\mathbb{D}(\eta, \underline{\nu}) = \frac{\prod_{e=1}^{K+1} \Gamma(\eta \cdot \nu_e)}{\Gamma(\eta \cdot)}, \sum_{e=1}^{K+1} \nu_e = 1. \tag{25}$$

Analogous to the above, a beta prior distribution is specified for the transformed parameter $b = e^{-\beta}$.

$$\Pi\{b \mid \gamma, \kappa\} = \frac{(b)^{\kappa \cdot \gamma - 1}(1-b)^{\kappa \cdot (1-\gamma) - 1}}{\mathbb{B}(\kappa \cdot \gamma, \kappa \cdot (1-\gamma))}. \tag{26}$$

The prior distribution of b is assumed independent of the prior distribution of $\underline{u}$.

Typically, to define the prior parameters, expert judgment concerning quantities of interest are elicited and equated to their theoretical expression for central tendency such as mean, median, or mode [see for example Cooke (1991)]. In addition, some quantification of the quality of the expert judgment is often given by specifying a variance or a probability interval for the prior quantity. Solving these equations generally leads to the desired parameter estimates. Specific quantities of interest for the problem at hand are the mission time reliabilities for each stress environment. An additional advantage of the Ordered Dirichlet distribution is that due to its mathematical properties, the incorporation of expert judgment is facilitated. From (24), for example, the prior marginal distribution for any u_e is obtained as a beta distribution given by

$$\Pi\{u_e\} = \frac{(u_e)^{\eta \cdot (1-\nu_{e\bullet}) - 1}(1-u_e)^{\eta \cdot \nu_{e\bullet} - 1}}{\mathbb{B}(\eta \cdot (1-\nu_{e\bullet}), \eta \cdot \nu_{e\bullet})}, \tag{27}$$

where $\mathbb{B}(\cdot, \cdot)$ is the well known beta constant. This distribution can be used to make prior probability statements concerning mission time reliabilities at the different stress levels due to the one-to-one relationships of these quantities to u_e. Specifically,

$$\{R(t) \mid u_e, \beta\} = Pr\{X > t \mid u_e, \beta\} = (u_e)^{\frac{t^\beta}{c}}, \tag{28}$$

where $\{R(t) \mid u_e, \beta\}$ is the reliability of a test item exposed to environment E_e for a mission time t given u_e.

To obtain the prior parameter values, estimates of prior mission time reliabilities must be obtained. The focus is on mission time reliabilities rather than failure rates, as these may more easily be obtained through elicitation methods focussing on observable quantities. Specifically, for a specified mission length, an estimate of R_e^* the mission time reliability in environment e, a quantile estimate R_ϵ^L for the mission reliability at use stress, and an estimate of R_ϵ° mission reliability after G mission time durations at use stress is required. Given this information, the following problem is solved numerically to obtain the prior parameter estimates (see van Dorp (1998) for details)

$$Solve\ \Theta = (c, \eta, \nu, \gamma, \kappa)\ from$$

$$1.\ Pr\{R_\epsilon(t) \leq R_\epsilon^* \mid \Theta\} = 0.50$$

$$2.\ Pr\{R_\epsilon(G \cdot t) \leq R_\epsilon^\circ \mid \Theta\} = 0.50$$

$$3.\ Pr\{R_\epsilon(t) \leq R_\epsilon^L \mid \Theta\} = 1 - q, q = 0.95$$

$$4.\ Pr\{R_e(t) \leq R_e^* \mid \Theta\} = 0.50,$$

$$e = 1, \ldots, K,\ e \neq \epsilon$$

Thus with the exception of the quantile estimate, all prior reliability estimates are treated as median values.

4 POSTERIOR APPROXIMATION

The expression for the likelihood given ALT data was derived in Section 2 and resulted in expressions for interval and type I censored data. Rather than performing prior-posterior analysis using these expressions, one may perform prior-posterior analysis by expressing likelihood in terms of $\underline{u}$ and b instead of $\underline{\lambda}$ and β using (22) and using well known properties of the Ordered Dirichlet distribution.

The posterior distribution follows for interval data and type I censored data by applying Bayes Theorem to the prior and the appropriate likelihood expressions. The derivation of the posterior distribution of is intractable in most cases. It is therefore suggested to use the well known Markov Chain Monte Carlo (MCMC) method approach (see, e.g., Casella and George (1992)). Through the MCMC approach, a sample of the posterior distribution can be obtained. From the sample, approximations of moments and an approximation of the joint posterior distribution may be derived. The approximations of marginal posterior distributions and, using (28), that of the mission time reliability at any stress level may be derived by the estimation of their quantiles. These quantiles may be estimated up to a desired level of accuracy using order statistics arguments (see, e.g., Mood et al (1974), pp. 513).

5 EXAMPLE

The following example is designed to show the flexibility of the Weibull ALT inference model. The use stress environment will be E_2 and different test items will be subjected to different step patterns. Assume that the following median mission time reliability estimates are available for a mission time of 1000 hours.

$R_1^* = 0.9$; $R_2^* = 0.8$; $R_3^* = 0.7$; $R_4^* = 0.6$; $R_5^* = 0.5$; $R_2^L = 0.4$; $R_2^G = 0.4$; $G = 2$,

An approximate solution to the prior parameters may be solved from the above data (see Van Dorp (1998)) yielding

$c = 2917399.332$; $\eta = 747.06$;
$\nu_1 = 0.2110$; $\nu_2 = 0.1833$; $\nu_3 = 0.1568$;
$\nu_4 = 0.1312$; $\nu_5 = 0.1065$; $\nu_6 = 0.2110$;
$\kappa = 472.74$, $\gamma = 0.1308$.

In this example, 6 proof-systems are available for testing. The step data concerning environments in each step and step interval times are specified for each testing stage, f, by the matrices A^f and T^f below

$$A^1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 \\ 5 & 4 & 3 & 2 & 1 \\ 1 & 3 & 5 & 4 & 2 \\ 1 & 3 & 5 & 4 & 2 \end{pmatrix},$$

$$T^1 = \begin{pmatrix} 100 & 100 & 100 & 100 & 100 \\ 100 & 100 & 100 & 100 & 100 \\ 100 & 100 & 100 & 100 & 100 \\ 100 & 100 & 100 & 100 & 100 \\ 100 & 100 & 100 & 100 & 100 \\ 100 & 100 & 100 & 100 & 100 \end{pmatrix},$$

$$A^2 = \begin{pmatrix} 1 & 3 & 5 & 4 & 2 \\ 1 & 3 & 5 & 4 & 2 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 \\ 5 & 4 & 3 & 2 & 1 \end{pmatrix},$$

$$T^2 = \begin{pmatrix} 100 & 100 & 100 & 100 & 100 \\ 100 & 100 & 100 & 100 & 100 \\ 100 & 100 & 100 & 100 & 100 \\ 100 & 100 & 100 & 100 & 100 \\ 100 & 100 & 100 & 100 & 100 \\ 100 & 100 & 100 & 100 & 100 \end{pmatrix}$$

In the second testing stage failed items from the first stage are assumed minimally repaired. Items that survive the first stage are continued on test in the second stage. The mission-time of the system was set to 1000 hours. The test results over the ALT are summarized in terms of $\underline{r}^f$, $\underline{q}^f$ in Table 1. Note that $q_j^f = 6$ indicates that the test item has survived the ALT stage without failure.

Table 1. ALT Test Data in Terms of $\underline{r}^f$, $\underline{q}^f$, $\underline{s}^f$.

Test Item		1	2	3	4	5	6
	r_j^1	495	500	295	500	395	500
ALT Stage 1	q_j^1	5	6	3	6	4	6
	r_j^2	295	95	500	195	500	295
ALT Stage 2	q_j^2	3	1	6	2	6	3

A prior-posterior analysis for both interval data and type I censored data is presented. The Gibbs Sampling Method was used to obtain posterior quantile estimates using test data obtained over 2 ALT stages for: (1) the scale parameters in each environment, and (2) the common shape parameter. The length of the Gibbs-Sequence generated was of length 100,000 and the Gibbs burn-in\Gibbs lag period was set to 25. MCMC diagnostics for this problem are discussed in Van Dorp and Mazzuchi (2003). Results are provided in Table 2.

Table 2. Prior & Posterior Parameter Estimates.

E_e	*Prior* λ_e^*	*Posterior* λ_e^*	*Posterior* λ_e
		Interval Data	*Type I Data*
1	$8.11 \cdot 10^{-8}$	$8.22 \cdot 10^{-8}$	$8.18 \cdot 10^{-8}$
.2	$1.72 \cdot 10^{-7}$	$1.73 \cdot 10^{-7}$	$1.73 \cdot 10^{-7}$
3	$2.75 \cdot 10^{-7}$	$2.77 \cdot 10^{-7}$	$2.76 \cdot 10^{-7}$
4	$3.93 \cdot 10^{-7}$	$3.95 \cdot 10^{-7}$	$3.95 \cdot 10^{-7}$
5	$5.34 \cdot 10^{-7}$	$5.36 \cdot 10^{-7}$	$5.36 \cdot 10^{-7}$
β^*	2.03	2.33	2.32

Distributional results may also be obtained. For example, Figures 4 and 5 covey the prior and posterior distribution for the shape and use stress scale parameter for the interval censoring case. Distributional results for the scale parameter or mission time reliability (for any specified mission time) at any stress level may also be generated.

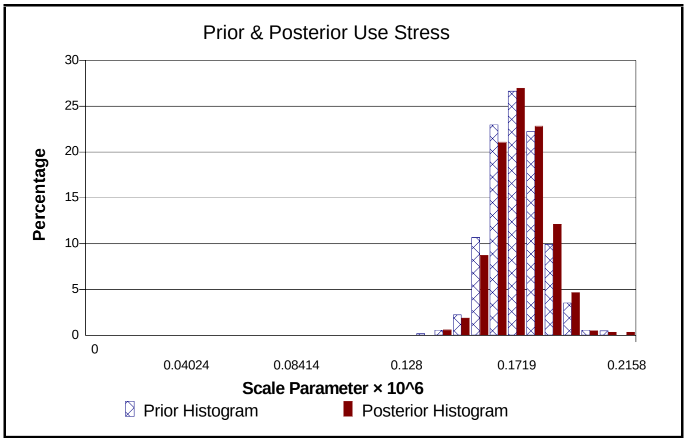

Figure 4. Prior & Posterior Scale Parameter for Environment 2 - Interval Data.

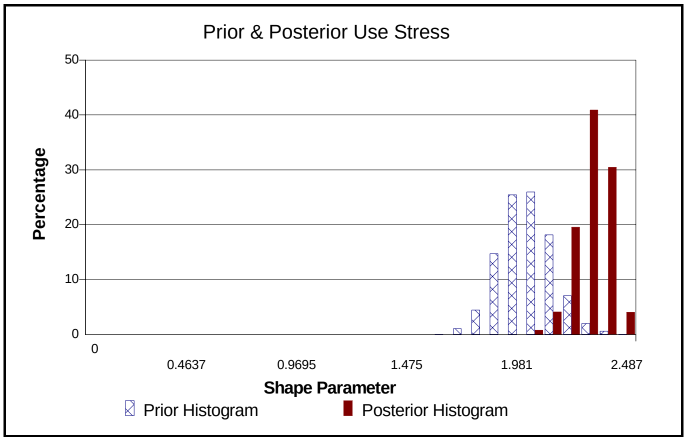

Figure 5. Prior & Posterior Shape Parameter - Interval Data.

It follows from Figures 4 and 5 that for this particular example, the greatest shift is observed in the distribution of the shape parameter rather than that of the scale parameter.

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