CHAPTER 16:
PORTFOLIO OPTIMIZATION USING SOLVER

PROBLEM DEFINITION:
Given a set of investments (for example stocks) how do we find a portfolio that has the lowest risk (i.e. lowest volatility or lowest variance) and yields an acceptable expected return?

- Harry Markowitz solved the above problem in 1950’s. For this work and other investment topics he received the Nobel Prize in Economics in 1991.

- Ideas discussed in this session are the basis for most methods of asset allocation used by Wall Street firms.
WEIGHTED SUM OF RANDOM VARIABLES

- A fixed number of N investments (or Stocks) are available.
- $S_i$: Random return during a calendar year on a dollar invested in investment $i$, where $i=1, \ldots, N$.

Examples:

- $s_i = 0.10$, dollar invested at the beginning of the years is worth $1.10$ at the end of the year
- $s_i = -0.20$, dollar invested at the beginning of the years is worth $0.80$ at the end of the year

Definition:

- $x_i$: the fraction of one dollar that is invested in investment $i$. Note that:
\[ \sum_{i=1}^{N} x_i = 1, \quad x_i \geq 0, i = 1, \ldots, N \]

- \( R \): be the random annual return on our investments. Then

\[ R = \sum_{i=1}^{n} x_i \times S_i \]

Note that:

- Because the returns on our investments \( S_i \) are random the annual return \( R \) on our portfolio of investments, defined by \((S_1, \ldots, S_N)\) and the specified weights \((x_1, \ldots, x_N)\) is random as well.
The expected (or average) annual return for R can be calculated as:

\[ E[R] = \sum_{i=1}^{n} x_i \cdot E[S_i] \]

**WHY?**

- We know that if \( X \) is a random variable, \( a \) and \( b \) are constants and \( Y = a \cdot X + b \) that:

\[ E[Y] = E[a \cdot X + b] = a \cdot E[X] + b \]

- We know that if \( X \) and \( Y \) are random variables and \( Z = X + Y \) that:

WHAT ABOUT THE UNCERTAINTY IN OUR ANNUAL RETURN R?

(or What about the variance of our Annual Return R?)

• We know that **if** the returns on the investments $S_i$ are **independent** random variables that

$$VAR[R] = \sum_{i=1}^{n} (x_i)^2 \cdot VAR[S_i]$$

**WHY?**

• We know that if $X$ is a random variable, $a$ and $b$ are constants and $Y=a*X+b$ that:

$$VAR[Y]=VAR[a*X+b]= a^2*VAR[X]$$
• We know that if $X$ and $Y$ are independent random variables and $Z=X+Y$ that:

$$\text{VAR}[Z] = \text{VAR}[X+Y] = \text{VAR}[X] + \text{VAR}[Y]$$

**BUT?**

Can we reasonably say that the annual return $S_i$ are independent random variables?

**Answer: NO!**

• We know that if the market (e.g. Dow Jones Index) is in an upward trend, that all stocks have a better chance of doing well.

• We know that if the market (e.g. Dow Jones Index) is in a downward trend, that all stocks have a higher chance of doing worse.
SO WHAT IS THE CORRECT FORMULA FOR THE VARIANCE OF THE ANNUAL RETURN $R$ WHEN THE INVESTMENTS ARE DEPENDENT?

INTERMEZZO: COVARIANCE

Let $X$, $Y$ be two random variables.

- If $X$ and $Y$ are independent there is no relationship between $X$ and $Y$, and:

$$\text{VAR}(X+Y)=\text{VAR}(X)+\text{VAR}(Y)$$
• If $X$ and $Y$ are **positively** dependent large values of $X$ tend to be associated with large values of $Y$

• If $X$ and $Y$ are **positively** dependent small values of $X$ tend to be associated with small values of $Y$

• If $X$ and $Y$ are **negatively** dependent large values of $X$ tend to be associated with small values of $Y$

• If $X$ and $Y$ are **negatively** dependent small values of $X$ tend to be associated with large values of $Y$.

**WHAT IS THE SIMPLEST RELATIONSHIP THAT YOU THINK OF BETWEEN $X$ AND $Y$?**

**ANSWER: A Linear Relationship!**
Let $X, Y$ be two random variables such that: $Y = aX + b$

- $Y = aX + b, a > 0$
  - Large Values of $X$ tend to be associated with Large Values of $Y$
  - Small Values of $X$ tend to be associated with Small Values of $Y$
  - **POSITIVE DEPENDENCE**

- $Y = aX + b, a < 0$
  - Large Values of $X$ tend to be associated with Small Values of $Y$
  - Small Values of $X$ tend to be associated with Small Values of $Y$
  - **NEGATIVE DEPENDENCE**
CONCLUSION:

- Covariance is positive when there are more (+,+ points and (-,-) points then (+,-) points and (-,+) points.
- Covariance is negative when there are more (+,-) points and (-,+) points then (+,+ points and (-,-) points.

THEREFORE:
Covariance is a measure of positive dependence or negative dependence.

If $Y=a*X+b$, what is the relationship between $\text{COV}(X,Y)$ and the slope of the line $a$?
• $E[Y] = a \cdot E[X] + b$

• $Y - E[Y] = a \cdot X + b - a \cdot E[X] - b = a \cdot (X - E[X])$

• $\text{Cov}(X, Y) = E[(Y - E[Y])(X - E[X])] = E[a \cdot (X - E[X])(X - E[X])] = a$
  
  $a \cdot E[(X - E[X])(X - E[X])] = a \cdot \text{Var}[X]$

  or

  $$a = \frac{\text{COV}(X, Y)}{\text{VAR}(X)}$$
**PERFECT LINEAR RELATIONSHIP**

\[ Y = a \times X + b, \ a > 0 \]

\[ a = \frac{\text{COV}(X,Y)}{\text{VAR}(X)} \]

**IMPERFECT LINEAR RELATIONSHIP**

\[ Y = \hat{a} \times X + \hat{b}, \ a > 0 \]

Best Fit Through Linear Regression

\[ \hat{a} = \frac{\text{COV}(X,Y)}{\text{VAR}(X)} \]

\[ = \frac{1}{n-1} \sum_{i=1}^{n} (y_i - \bar{y})(x_i - \bar{x}) \]

\[ = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})(x_i - \bar{x}) \]
CONCLUSION:

- If you have a set of points \((x_i, y_i), i=1,\ldots,n\) you can estimate positive
dependence or negative dependence by calculating \(\text{COV}(X,Y)\).

- If you have a set of points \((x_i, y_i), i=1,\ldots,n\) you can estimate the slope of the
“linear relationship” by calculating \(\hat{a}\).

HOW CAN WE DISTINGUISH BETWEEN A PERFECT LINEAR
RELATIONSHIP AND AN IMPERFECT LINEAR RELATIONSHIP
GIVEN THE SET OF POINTS \((x_i, y_i), i=1,\ldots,n\)?
Let $X,Y$ be two random variables such that

\[ Y = aX + b, \]

- $\text{Cov}(X,Y) = a \cdot \text{Var}[X]$
- $\text{Var}[Y] = a^2 \cdot \text{Var}[X]$

**CORRELATION BETWEEN TWO RANDOM VARIABLES**

\[
\text{Cor}(X,Y) = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X) \cdot \text{Var}(Y)}} = \frac{a \cdot \text{Var}(X)}{\sqrt{\text{Var}(X) \cdot a^2 \cdot \text{Var}(X)}} = \pm 1
\]
### Perfect Linear Relationship

- Equation: \( Y = aX + b, \ a > 0 \)
- Graph: Linear relationship with \( a \) positive.
- Correlation: \( \text{COR}(X,Y) = 1 \)

### Imperfect Linear Relationship

- Equation: \( Y = aX + b, \ a > 0 \)
- Graph: Linear relationship but less perfect.
- Correlation: \( \text{COR}(X,Y) < 1 \)

- Slope: \( \hat{a} = \frac{\hat{\text{COV}}(X,Y)}{\hat{\text{VAR}}(X)} \)
- Condition: \( \hat{\text{VAR}}(Y) > a^2 \hat{\text{VAR}}(X) \)
• If X and Y are dependent random variables

\[ \text{VAR}(X+Y) = \text{VAR}(X) + \text{VAR}(Y) + 2 \cdot \text{COV}(X,Y) \]

CONCLUSION:

• If X, Y are positive dependent variables the level of variation of X+Y is amplified by the dependency.

• If X, Y are negative dependent variables the level of variation of X+Y is reduced by the dependency (the random variables tend to “cancel each other out”).

BACK TO OUR ORIGINAL PROBLEM
• \( S_i \): Random returns during a calendar year on a dollar invested in investment I, where \( i=1, \ldots, N \).

• Random returns \( S_i \) are \textbf{dependent} random variables

• \( x_i \): the fraction of one dollar that is invested in investment \( i \). Note that:

\[
\sum_{i=1}^{N} x_i = 1, \quad x_i \geq 0, i = 1, \ldots, N
\]

• \( R \) be the random annual return on our investments. Then

\[
R = \sum_{i=1}^{n} x_i \ast S_i
\]
The expected (or average) annual return for R can be calculated as:

\[
E[R] = \sum_{i=1}^{n} x_i * E[S_i]
\]

The variance of the annual return for R can be calculated as:

\[
Var[R] = \sum_{i=1}^{n} x_i^2 * Var[S_i] + \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} x_i x_j Cov(S_i, S_j)
\]

or

\[
Var[R] = \sum_{i=1}^{n} x_i^2 * Var[S_i] + \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} x_i x_j Cor(S_i, S_j) \sqrt{Var[S_i] Var[S_j]}
\]

EQUATIONS LOOK COMPLICATED!
INTERMEZZO: MATRIX - VECTOR MULTIPLICATION

M·N matrix:

\[
A = \begin{pmatrix}
    a_{1,1} & a_{1,2} & \cdots & a_{1,N} \\
    a_{2,1} & a_{2,2} & \ddots & \vdots \\
    \vdots & \vdots & \ddots & a_{M-1,N} \\
    a_{M,1} & \cdots & a_{M,N-1} & a_{m,n}
\end{pmatrix}
\]

M rows, N Columns

N-Vector (or N·1 matrix):

\[
\underline{x} = \begin{pmatrix}
x_1 \\
\vdots \\
x_N
\end{pmatrix}
\]
Matrix-Vector Product:

\[
Ax = \begin{pmatrix}
    a_{1,1} & a_{1,2} & \cdots & a_{1,N} \\
    a_{2,1} & a_{2,2} & \cdots & \vdots \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{M,1} & \cdots & a_{M,N-1} & a_{m,n}
\end{pmatrix}
\begin{pmatrix}
    x_1 \\
    \vdots \\
    x_N
\end{pmatrix}
= \begin{pmatrix}
    \sum_{j=1}^{N} a_{1,j} x_j \\
    \sum_{j=1}^{N} a_{2,j} x_j \\
    \vdots \\
    \sum_{j=1}^{N} a_{M,j} x_j
\end{pmatrix}
= M \cdot \text{Vector}
\]

\[(M \cdot N \cdot \text{Matrix}) \cdot (N \cdot \text{vector}) = M \cdot \text{Vector}\]

\[(M \cdot N \cdot \text{Matrix}) \cdot (N \cdot 1 \cdot \text{Matrix}) = M \cdot 1 \cdot \text{Matrix}\]
EXAMPLE:

\[
\begin{pmatrix}
1 & 2 & 3 \\
2 & 4 & 5
\end{pmatrix}
\begin{pmatrix}
2 \\
3 \\
4
\end{pmatrix}
= \begin{pmatrix}
1 \times 2 + 2 \times 3 + 3 \times 4 \\
2 \times 2 + 4 \times 3 + 5 \times 4
\end{pmatrix}
= \begin{pmatrix}
20 \\
36
\end{pmatrix}
\]

VECTOR - MATRIX MULTIPLICATION

M·N matrix:

\[
A = \begin{pmatrix}
a_{1,1} & a_{1,2} & \cdots & a_{1,N} \\
a_{2,1} & a_{2,2} & \cdots & \vdots \\
\vdots & \vdots & \ddots & a_{M-1,N} \\
a_{M,1} & \cdots & a_{M,N-1} & a_{m,n}
\end{pmatrix}
\]

M rows, N Columns
TRANSPOSED M-Vector (or M\cdot1 matrix) :

\[ \mathbf{x}^T = \left( x_1 \cdots x_M \right) \]

Vector - Matrix Product:

\[ \mathbf{x}^T \mathbf{A} = \left( x_1 \cdots x_M \right) \left( \begin{array}{cccc} a_{1,1} & a_{1,2} & \cdots & a_{1,N} \\ a_{2,1} & a_{2,2} & \cdots & \vdots \\ \vdots & \vdots & \ddots & a_{M-1,N} \\ a_{M,1} & \cdots & a_{M,N-1} & a_{m,n} \end{array} \right) = \left( \begin{array}{cccc} \sum_{j=1}^{N} x_j a_{j,1} & \sum_{j=1}^{N} x_j a_{j,2} & \cdots & \sum_{j=1}^{N} x_j a_{j,N} \end{array} \right) \]

\((\text{Transposed M – Vector}) \cdot (\text{M\cdotN-Matrix}) = \text{N-Vector}\)

\((1\cdot\text{M-Matrix}) \cdot (\text{M\cdotN-Matrix}) = 1\cdot\text{N-Matrix}\)
EXAMPLE:

\[
\begin{pmatrix} 4 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \end{pmatrix} = \\
\begin{pmatrix} 4 \times 1 + 5 \times 2 & 4 \times 2 + 5 \times 4 & 4 \times 3 + 5 \times 5 \end{pmatrix} = \\
\begin{pmatrix} 14 & 28 & 37 \end{pmatrix}
\]
MATRIX – MATRIX MULTIPLICATION: $(M \cdot N \text{-Matrix}) \times (N \cdot P \text{-Matrix}) = M \cdot P \text{-Matrix}$

$$AB = \begin{pmatrix}
a_{1,1} & a_{1,2} & \cdots & a_{1,N} \\
a_{2,1} & a_{2,2} & & \vdots \\
& & \ddots & a_{M-1,N} \\
a_{M,1} & \cdots & a_{M,N-1} & a_{M,N}
\end{pmatrix} \begin{pmatrix}
b_{1,1} & b_{1,2} & \cdots & b_{1,P} \\
b_{2,1} & b_{2,2} & & \vdots \\
& & \ddots & b_{N-1,P} \\
b_{N,1} & \cdots & b_{N,P-1} & b_{N,P}
\end{pmatrix}$$

$$= \begin{pmatrix}
\sum_{j=1}^{N} a_{1,j} b_{j,1} & \sum_{j=1}^{N} a_{1,j} b_{j,2} & \cdots & \sum_{j=1}^{N} a_{1,j} b_{j,P} \\
\sum_{j=1}^{N} a_{2,j} b_{j,1} & \sum_{j=1}^{N} a_{2,j} b_{j,2} & & \vdots \\
& & \ddots & \sum_{j=1}^{N} a_{2,j} b_{j,P} \\
\sum_{j=1}^{N} a_{M,j} b_{j,1} & \cdots & \sum_{j=1}^{N} a_{M,j} b_{j,P-1} & \sum_{j=1}^{N} a_{M,j} b_{j,P}
\end{pmatrix}$$
EXAMPLE:

\[
\begin{pmatrix}
1 & 2 & 3 \\
2 & 4 & 5 \\
\end{pmatrix}
\begin{pmatrix}
1 & 2 \\
3 & 4 \\
5 & 6 \\
\end{pmatrix}
= \\
\begin{pmatrix}
1*1 + 2*3 + 3*5 & 1*2 + 2*4 + 3*6 \\
2*1 + 4*3 + 5*5 & 2*2 + 4*4 + 5*6 \\
\end{pmatrix}
= \\
\begin{pmatrix}
22 & 28 \\
39 & 50 \\
\end{pmatrix}
\]

NOTE THAT:

\[(M \cdot N \cdot \text{Matrix}) \ast (N \cdot P \cdot \text{Matrix}) = M \cdot P \cdot \text{Matrix}\]
BACK TO OUR ORIGINAL PROBLEM:

\[ Var[R] = \sum_{i=1}^{n} x_i^2 \cdot Var[S_i] + \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} x_i x_j \cdot Cor(S_i, S_j) \cdot \sqrt{Var[S_i] \cdot Var[S_j]} \]

HOWEVER INTRODUCING WEIGHT VECTOR:

\[ \underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} \]

AND TRANSPOSE OF WEIGHT VECTOR;

\[ \underline{x}^T = \begin{pmatrix} x_1^T \\ \vdots \\ x_N \end{pmatrix} = (x_1, \ldots, x_n) \]
AND VARIANCE – COVARIANCE MATRIX:

\[
\begin{pmatrix}
\text{Cov}(S_1, S_1) & \text{Cov}(S_1, S_2) & \cdots & \text{Cov}(S_1, S_N) \\
\text{Cov}(S_2, S_1) & \text{Cov}(S_2, S_2) & \cdots & \vdots \\
\vdots & \vdots & \ddots & \text{Cov}(S_{N-1}, S_N) \\
\text{Cov}(S_N, S_1) & \cdots & \text{Cov}(S_{N-1}, S_N) & \text{Cov}(S_N, S_N)
\end{pmatrix}
\]

AND REALIZING THAT: \( \text{COV}(S_i, S_i) = \text{VAR}(S_i) \)

\[
\text{Var}[R] = \sum_{i=1}^{n} x_i^2 \ast \text{Var}[S_i] + \\
+ \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} x_i x_j \text{Cor}(S_i, S_j) \sqrt{\text{Var}[S_i] \text{Var}[S_j]} = \mathbf{x}^T \sum \mathbf{x}
\]

HOMEWORK: SHOW THAT ABOVE ASSERTION IS CORRECT!
PROBLEM DEFINITION:
Given a set of investments (for example stocks) how do we find a portfolio that has the lowest risk (i.e. lowest volatility or lowest variance) and yields an acceptable expected return?

- $S_i$: Random return during a calendar year on a dollar invested in investment $i$, where $i=1, \ldots, N$.

- $x_i$: the fraction of one dollar that is invested in investment $i$. Note that:

$$\sum_{i=1}^{N} x_i = 1, \quad x_i \geq 0, \quad i = 1, \ldots, N$$
• The expected (or average) annual return for R can be calculated as:

\[ E[R] = \sum_{i=1}^{n} x_i \cdot E[S_i] = g(x) \]

• The variance of the annual return for R can be calculated as:

\[ Var[R] = x^T \sum x = f(x) \]

**OPTIMIZATION PROBLEM:**

Minimize: \[ f(x) \]

Subject to: \[ g(x) \geq r\% \]

\[ \sum_{i=1}^{N} x_i = 1 \quad x_i \geq 0, i = 1, \ldots, N \]
WHAT TYPE OF OPTIMIZATION PROBLEM IS THIS?

- Variance – Covariance Matrix $\Sigma$ is known to be positive definite.

- Objective function is a multidimensional quadratic function. Hessian Matrix of $x^T \Sigma x$ is $\Sigma$.

- Conclusion: Objective Function is Convex

- Constraint function $g(x) - r^\%$ linear function (and thus convex)

- Constraint function $\sum_{i=1}^{N} x_i = 1$ can be rewritten as:

$$\sum_{i=1}^{N} x_i \geq 1, \sum_{i=1}^{N} x_i \leq 1$$

both of which are linear function (and thus convex).
CONCLUSION: OPTIMIZATION PROBLEM IS CONVEX OPTIMIZATION PROBLEM AND HAS A UNIQUE GLOBAL OPTIMUM.

EXAMPLE:

<table>
<thead>
<tr>
<th></th>
<th>Stock 1</th>
<th>Stock 2</th>
<th>Stock 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean return</td>
<td>0.14</td>
<td>0.11</td>
<td>0.1</td>
</tr>
<tr>
<td>StDev of return</td>
<td>0.2</td>
<td>0.15</td>
<td>0.08</td>
</tr>
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</table>

<table>
<thead>
<tr>
<th></th>
<th>Stock 1</th>
<th>Stock 2</th>
<th>Stock 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Correlations</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Stock 1</td>
<td>1</td>
<td>0.6</td>
<td>0.4</td>
</tr>
<tr>
<td>Stock 2</td>
<td>0.6</td>
<td>1</td>
<td>0.7</td>
</tr>
<tr>
<td>Stock 3</td>
<td>0.4</td>
<td>0.7</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Stock 1</th>
<th>Stock 2</th>
<th>Stock 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Covariances</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Stock 1</td>
<td>0.040</td>
<td>0.018</td>
<td>0.006</td>
</tr>
<tr>
<td>Stock 2</td>
<td>0.018</td>
<td>0.023</td>
<td>0.008</td>
</tr>
<tr>
<td>Stock 3</td>
<td>0.006</td>
<td>0.008</td>
<td>0.006</td>
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### AFTER OPTIMIZATION

<table>
<thead>
<tr>
<th>Investment decisions</th>
<th>Stock 1</th>
<th>Stock 2</th>
<th>Stock 3</th>
<th>Total</th>
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<tbody>
<tr>
<td>Fractions to invest</td>
<td>1/2</td>
<td>0</td>
<td>0</td>
<td>1/2</td>
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</tbody>
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<table>
<thead>
<tr>
<th>Constraint on expected portfolio return</th>
</tr>
</thead>
<tbody>
<tr>
<td>Actual</td>
</tr>
<tr>
<td>---------</td>
</tr>
<tr>
<td>0.1200</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Portfolio variance</th>
<th>Portfolio stddev</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0148</td>
<td>0.1217</td>
</tr>
</tbody>
</table>
SUPPOSE WE INCREASE THE REQUIRED EXPECTED RETURN, WHAT HAPPENS TO THE STANDARD DEVIATION OF THE PORTFOLIO?

<table>
<thead>
<tr>
<th>Required Return</th>
<th>Stock 1</th>
<th>Stock 2</th>
<th>Stock 3</th>
<th>Port stddev</th>
<th>Exp return</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.100</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0.0800</td>
<td>0.100</td>
</tr>
<tr>
<td>0.105</td>
<td>1/8</td>
<td>0</td>
<td>7/8</td>
<td>0.0832</td>
<td>0.105</td>
</tr>
<tr>
<td>0.110</td>
<td>1/4</td>
<td>0</td>
<td>3/4</td>
<td>0.0922</td>
<td>0.110</td>
</tr>
<tr>
<td>0.115</td>
<td>3/8</td>
<td>0</td>
<td>5/8</td>
<td>0.1055</td>
<td>0.115</td>
</tr>
<tr>
<td>0.120</td>
<td>1/2</td>
<td>0</td>
<td>1/2</td>
<td>0.1217</td>
<td>0.120</td>
</tr>
<tr>
<td>0.125</td>
<td>5/8</td>
<td>0</td>
<td>3/8</td>
<td>0.1397</td>
<td>0.125</td>
</tr>
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<td>0.130</td>
<td>3/4</td>
<td>0</td>
<td>1/4</td>
<td>0.1591</td>
<td>0.130</td>
</tr>
<tr>
<td>0.135</td>
<td>7/8</td>
<td>0</td>
<td>1/8</td>
<td>0.1792</td>
<td>0.135</td>
</tr>
<tr>
<td>0.140</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0.2000</td>
<td>0.140</td>
</tr>
</tbody>
</table>
THE EFFICIENT FRONTIER

Efficient Frontier (Risk Versus Expected Return)

Expected Return

Standard Deviation

0.0500
0.1000
0.1500
0.2000

0.050
0.100
0.150
0.200

0.0500
0.1000
0.1500
0.2000

0.050
0.100
0.150
0.200

Expected Return