CHAPTER 13: DETERMINING MONTHLY LOAN PAYMENTS

Suppose you are planning to buy a new car. The car costs $25000, you have been approved for financing the loan with a 5 year term through the car dealer and the monthly interest rate on the loan is 1%. What are the monthly payments that you will have to make? Excel has a standard function PMT(rate, number of periods, loan amounts) to calculate these payments.

Length of loan 60  
Interest rate/month $ 0.01  
Principal (Loan Amount) $ 25,000.00  

Monthly payment ($556.11)

HOW IS THIS MONTHLY PAYMENT CALCULATED?
Second Method - Problem Formulation

Define:

\[ B(k-1) = \text{Ending Loan Balance of Month (k-1)} \]

Remaining loan balance at the end of Month k is calculated by reducing the remaining loan balance at the end of month (k-1) by:

\[
\text{Monthly Payment} - \text{Interest on the remaining balance at the end of Month (k-1)}
\]

**Interpretation:**

Some of the monthly payment is used to pay interest. What is left over is used to reduce the loan amount.

Hence:

\[ B(k) = B(k-1) - (\text{MP} - B(k-1)\times\text{IR})) = B(k-1)(1+\text{IR}) - \text{MP} \]
But:

\[ B(k-1) = B(k-2)(1+IR) - MP \]

Therefore:

\[ B(k) = \{B(k-2)(1+IR) - MP\} (1+IR) - MP \]

\[ = B(k-2)(1+IR)^2 - MP - MP(1+IR) \]

**Conclusion:**

\[
B(k) = B(i) \cdot (1 + IR)^{k-i} - MP \sum_{l=0}^{k-i-1} (1 + IR)^l
\]

\[
MP = \frac{B(i) \cdot (1 + IR)^{k-i} - B(k)}{\sum_{l=0}^{k-i-1} (1 + IR)^l}
\]
INTERMEZZO:

\[ S(x, n) = \sum_{l=0}^{n} x^l \iff \]

\[ xS(x, n) = \sum_{l=0}^{n} x^{l+1} = \sum_{l=1}^{n+1} x^l \iff \]

\[ xS(x, n) = x^{n+1} - 1 + \sum_{l=0}^{n} x^l = x^{n+1} - 1 + S(x, n) \iff \]

\[ (x - 1)S(x, n) = x^{n+1} - 1 \iff \]

\[ S(x, n) = \frac{x^{n+1} - 1}{x - 1} \]
Setting: \( x = (1 + IR) \), \( n = k - i - 1 \)

\[
\sum_{l=0}^{k-i-1} (1 + IR)^l = S(1 + IR, k - i - 1) \iff \\
= \frac{(1 + IR)^{k-i-1+1} - 1}{(1 + IR) - 1} = \frac{(1 + IR)^{k-i} - 1}{IR}
\]

Hence:

\[
MP = \frac{B(i) \cdot (1 + IR)^{k-i} - B(k)}{\sum_{l=0}^{k-i-1} (1 + IR)^l}
\]

\[
MP = \frac{B(i) \cdot IR \cdot (1 + IR)^{k-i} - IR \cdot B(k)}{(1 + IR)^{k-i} - 1}
\]
SOLVING FOR MONTHLY PAYMENT BY SETTING
THE LOAN BALANCE AT THE END OF 60 MONTHS
EQUAL TO ZERO WE CONCLUDE THAT:

\[ k = 60, \ B(60) = 0, \ i = 0, \ B(0) = \$25000, \ IR = 1\% \]

\[
MP = \frac{\$25000 \cdot 0.01 \cdot (1 + 0.01)^{60}}{(1 + 0.01)^{60} - 1} = 556.11
\]
Third Method – Using Solver

Remember:

\[ B(k-1) = \text{Ending Loan Balance of Month } (k-1) \]

Remaining loan balance at the end of Month k is calculated by reducing the remaining loan balance at the end of month (k-1) by:

\[ \text{Monthly Payment} – \text{Interest on the remaining balance at the end of Month } (k-1) \]

**Interpretation:**
Some of the monthly payment is used to pay interest. What is left over is used to reduce the loan amount.

**Hence:**

\[ B(k) = B(k-1) – (\text{MP} - B(k-1) \times IR)) \]
Method 3

Monthly Payment? $ 500.00

<table>
<thead>
<tr>
<th>Month</th>
<th>Beginning Balance</th>
<th>Payment</th>
<th>Interest</th>
<th>Ending Balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$25,000.00</td>
<td>$500.00</td>
<td>$250.00</td>
<td>$24,750.00</td>
</tr>
<tr>
<td>2</td>
<td>$24,750.00</td>
<td>$500.00</td>
<td>$247.50</td>
<td>$24,497.50</td>
</tr>
<tr>
<td>3</td>
<td>$24,497.50</td>
<td>$500.00</td>
<td>$244.98</td>
<td>$24,242.48</td>
</tr>
<tr>
<td>4</td>
<td>$24,242.48</td>
<td>$500.00</td>
<td>$242.42</td>
<td>$23,984.90</td>
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<tr>
<td>5</td>
<td>$23,984.90</td>
<td>$500.00</td>
<td>$239.85</td>
<td>$23,724.75</td>
</tr>
<tr>
<td>10</td>
<td>$22,657.87</td>
<td>$500.00</td>
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<tr>
<td>57</td>
<td>$6,354.75</td>
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<td>$63.55</td>
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<td>58</td>
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<td>$59.18</td>
<td>$5,477.49</td>
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<tr>
<td>59</td>
<td>$5,477.49</td>
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<td>$54.77</td>
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<tr>
<td>60</td>
<td>$5,032.26</td>
<td>$500.00</td>
<td>$50.32</td>
<td>$4,582.58</td>
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</table>
Hence, setting the Ending Balance (ROW 60) equal to zero by changing the Monthly Payment yields the Monthly Payment. Using Solver we obtain as a solution to the problem.

**Method 3**

**Monthly Payment? $ 556.11**

<table>
<thead>
<tr>
<th>Month</th>
<th>Beginning Balance</th>
<th>Payment</th>
<th>Interest</th>
<th>Ending Balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$25,000.00</td>
<td>$556.11</td>
<td>$250.00</td>
<td>$24,693.89</td>
</tr>
<tr>
<td>2</td>
<td>$24,693.89</td>
<td>$556.11</td>
<td>$246.94</td>
<td>$24,384.72</td>
</tr>
<tr>
<td>59</td>
<td>$1,095.76</td>
<td>$556.11</td>
<td>$10.96</td>
<td>$550.61</td>
</tr>
<tr>
<td>60</td>
<td>$550.61</td>
<td>$556.11</td>
<td>$5.51</td>
<td>(0.00)</td>
</tr>
</tbody>
</table>
HOMEWORK:

Derive the function

$$MP = \frac{B(0) \cdot IR \cdot (1 + IR)^n}{(1 + IR)^n - 1}$$

by directly equating the net present value of cash-flow of $n$ payments of size $MP$ equal to the loan amount $B(0)$

Hint: Use the relationship derived in these notes

$$S(x, n) = \frac{x^{n+1} - 1}{x - 1}$$