Solutions

8.1. Answers will vary considerably, but students might think about subjective probability as a degree of belief, uncertainty in one's mind, or a willingness to bet or accept lotteries. They may appropriately contrast the subjective interpretation of probability with a frequency interpretation.

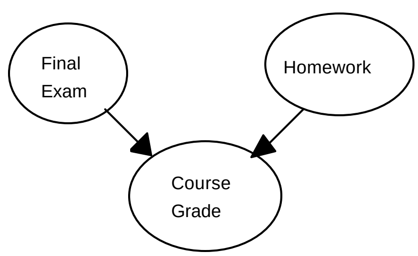
8.2. The model under discussion relates a number of financial ratios to the probability of default. First, there is a good deal of subjective judgment involved in deciding which financial ratios to include in such a model. Second, in using past data your friend has made the implicit assumption that no fundamental changes in causes of default will occur, or that the data from the past are appropriate for understanding which firms may default in the future. A bank officer using this model is making an implicit subjective judgment that the model and data used are adequate for estimating the probability of default of the particular firms that have applied for loans. Finally, the bank officer implicitly judges that your friend has done a good job!

8.3. Assessing a discrete probability requires only one judgment. Assessing a continuous probability distribution can require many subjective judgments of interval probabilities, cumulative probabilities, or fractiles in order to sketch out the CDF. Even so, the fundamental probability assessments required in the continuous case are essentially the same as in the discrete case.

8.4. Answers will, of course, vary a lot. As a motivation for careful quantitative modeling of probability, it is instructive to collect responses from a number of people in the class and show the ranges of their responses. Thus, it is clear that different people interpret these verbal phrases in different ways.

Answers can be checked for consistency, as well. In particular, a > 0.5, g > 0.5, l < j, e < j, p < m < i, and o < f < k.

8.5. Answers will vary here, too, but many students will decompose the assessment into how well they will do on homework (for which they have a good deal of information) and how well they will do on a final exam or project.



8.6. The students' assessments may vary considerably. However, they should be reasonable and indicate in some way that some effort went into the assessment process. Answers should include some discussion of thought processes that led to the different assessments.

8.7. It is possible to assess probabilities regarding one's own performance, but such assessments are complicated because the outcome depends on effort expended. For example, an individual might assess a relatively high probability for an A in a course, thus creating something of a personal commitment to work hard in the course.

8.8. Considering the assessments in 8.8, it might be appropriate to decompose the event into uncertain factors (homework, professor's style, exams, and so on) and then think about how much effort to put into the course. It would then be possible to construct risk profiles for the final grade, given different levels of effort.

8.9. This problem calls for the subjective assessment of odds. Unfortunately, no formal method for assessing odds directly has been provided in the text. Such a formal approach could be constructed in terms of bets or lotteries as in the case of probabilities, but with the uncertainty stated in odds form.

8.10. First, let us adopt some notation. Let NW and NL denote "Napoleon wins" and "Napoleon loses," respectively. Also, let "P&E" denote that the Prussians and English have joined forces.
The best way to handle this problem is to express Bayes' theorem in odds form. Show that

$$\frac{P(NW \mid P\&E)}{P(NL \mid P\&E)} = \frac{P(P\&E \mid NW)}{P(P\&E \mid NL)} \frac{P(NW)}{P(NL)} .$$

We have that P(NW) = 0.90 and P(NW | P&E) = 0.60, and so P(NL) = 0.10 and P(NL | P&E) = 0.40. Thus, we can substitute:

$$\frac{0.60}{0.40} = \frac{P(P\&E \mid NW)}{P(P\&E \mid NL)} \frac{0.90}{0.10}$$

or

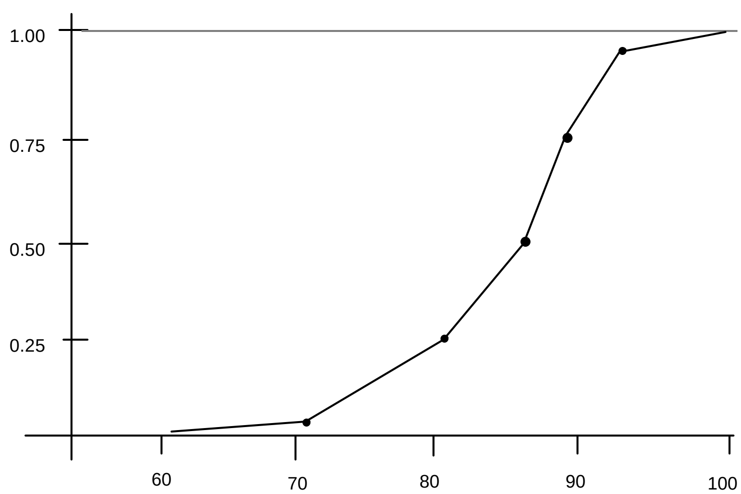
$$\frac{P(P\&E \mid NW)}{P(P\&E \mid NL)} = \frac{1}{6} .$$

Thus, Napoleon would have had to judge the probability of the Prussian and English joining forces as six time more likely if he is to lose than if he is to win.

8.11. This problem requires a student to assess a subjective CDF for his or her score in the decision analysis course. Thus, answers will vary considerably, depending on personal assessments. For illustrative purposes, assume that the student assesses the 0.05, 0.25, 0.50, 0.75, and 0.95 fractiles:

$x_{0.05} = 71$ $x_{0.25} = 81$ $x_{0.50} = 0.87$ $x_{0.75} = 89$ $x_{0.95} = 93$

These assessments can be used to create a subjective CDF:



To use these judgments in deciding whether to drop the course, we can use either bracket medians or the Pearson-Tukey method. The Pearson-Tukey method approximates the expected DA score as:

E_{P-T}(DA Score) ≈ 0.185 (71) + 0.63 (87) + 0.185 (93) = 85.15.

Assume that the student has a GPA of 2.7. Using E_{P-T}(DA Score) = 85.15 to calculate expected salary,

E(Salary | Drop Course) = $4000 (2.7) + $16,000 = $26,800

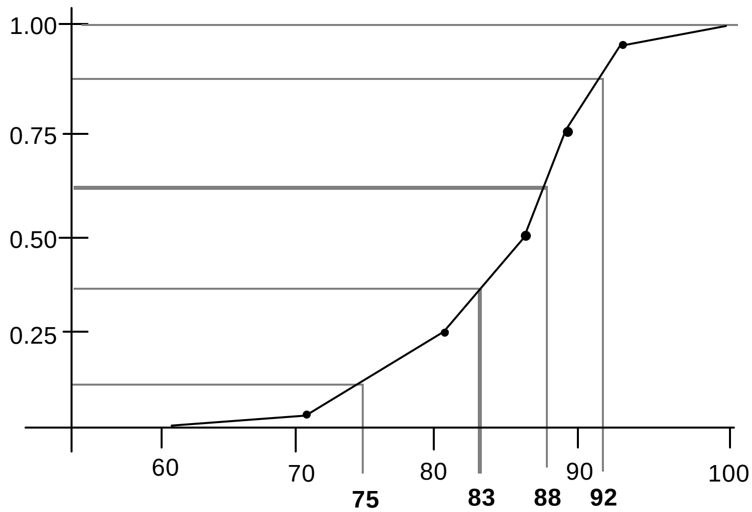
$E(\text{Salary} \mid \text{Don't drop}) = 0.6\ (\$4000 \text{ x } 2.7) + 0.4\ (\$170 \times E_{P\text{-}T}(\text{DA Score})) + \$16,000$

$= 0.6\ (\$4000 \times 2.7) + 0.4\ (\$170 \times 85.15) + \$16,000$

$= \$28,270.$

Thus, the optimal choice is not to drop the course.

To use the bracket median approach, we first determine that bracket medians for four equal-probability intervals would be approximately 75, 83, 88, and 92.



Thus, the bracket-median approximation would be

$E_{BM}(\text{DA Score}) \approx 0.25\ (75) + 0.25\ (83) + 0.25\ (88) + 0.25\ (92) = 84.5$

Using this in calculating expected salaries:

$E(\text{Salary} \mid \text{Drop Course}) = \$4000\ (2.7) + \$16,000 = \$26,800$

$E(\text{Salary} \mid \text{Don't drop}) = 0.6\ (\$4000 \times 2.7) + 0.4\ (\$170 \times E_{BM}(\text{DA Score})) + \$16,000$

$= 0.6\ (\$4000 \times 2.7) + 0.4\ (\$170 \times 84.5) + \$16,000$

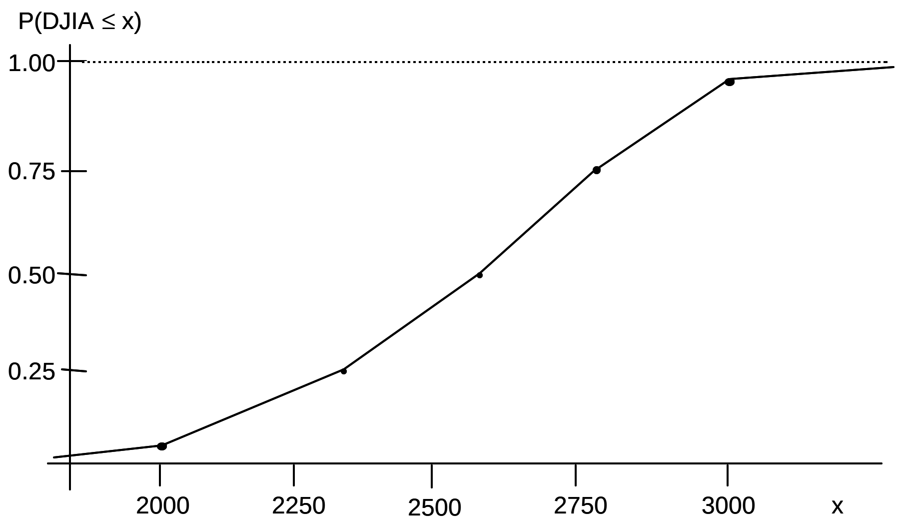
$= \$28,226.$

Again, the conclusion is not to drop the course.

8.12. Again, the assessments will be based on personal judgments and will vary among students. As an example, suppose the following assessments are made:

$P(\text{DJIA} \leq 2000) = 0.05$
$P(\text{DJIA} > 3000) = 0.05$
$P(\text{DJIA} \leq 2600) = 0.50$
$P(\text{DJIA} \leq 2350) = 0.25$
$P(\text{DJIA} \leq 2800) = 0.75$

These assessments result in the following graph:

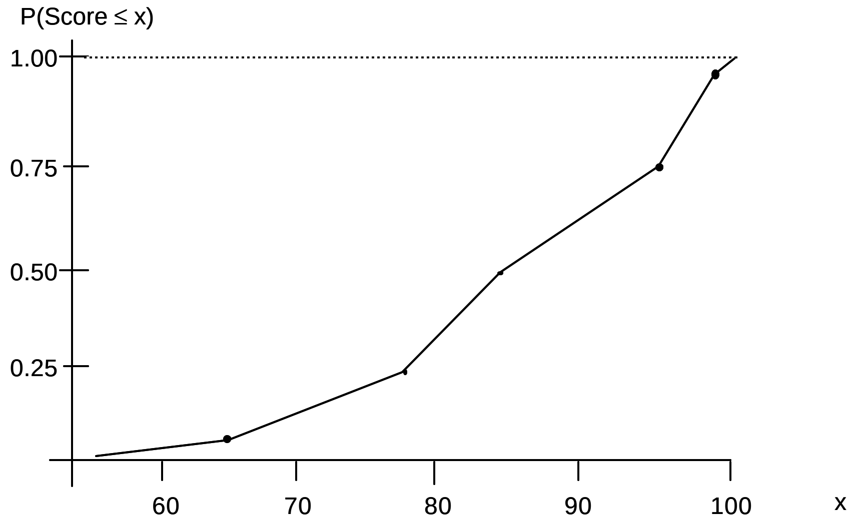


8.13. If you have worked this problem and looked in the back of the book for the answers, you have most likely found that relatively few of the answers fell within the ranges you stated, indicating that you, like most people are very overconfident in your judgments. Given this, it would make sense to return to Problem 8.12 and broaden the assessed distributions. (Note from Dr. Clemen: I have tried to do exercises like this myself, knowing about the overconfidence phenomenon and how to make subjective probability judgments, and I still make the same mistake!)

8.14. The cumulative distribution function provides a "picture" of what the forecaster sees as reasonable outcomes for the uncertain quantity. If the CDF is translated into a probability density function, it is still easier to see how the forecaster thinks about the relative chances of the possible outcomes.
The key advantages of probabilistic forecasting are 1) that it provides a complete picture of the uncertainty, as opposed to a point forecast which may give no indication of how accurate it is likely to be or how large the error might be; and 2) the decision maker can use the probability distribution in a decision analysis if desired. The disadvantage is that making the necessary assessments for the probabilistic forecast may take some time. Some decision makers (the uninitiated) may have difficulty interpreting a probabilistic forecast.

8.15. a. This question requires students to make personal judgments. As an example, suppose the following assessments are made:

$P(S \leq 65) = 0.05$
$P(S > 99) = 0.05$
$P(S \leq 78) = 0.25$
$P(S \leq 85) = 0.50$
$P(S \leq 96) = 0.75$

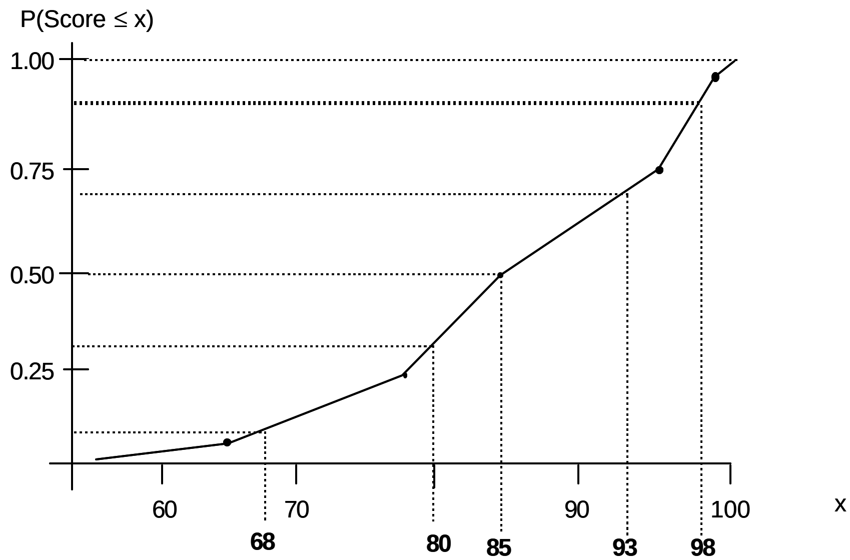


b. Now the student would ask whether she would be willing to place a 50-50 bet in which she wins if $78 < S \leq 85$ and loses if $85 < S \leq 96$. Is there a problem with betting on an event over which you have some control? See problems 8.8, 8.9.

c. Three-point Pearson-Tukey approximation:

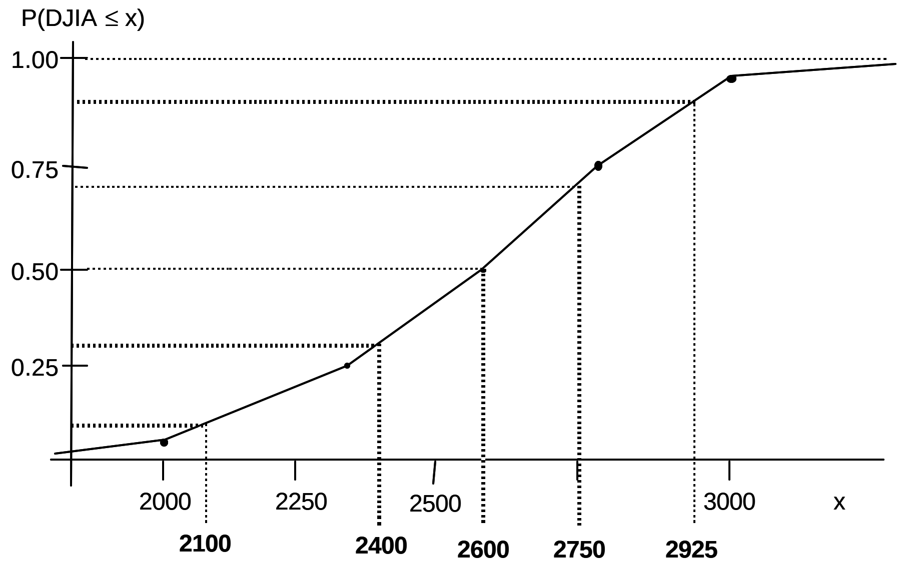
$E_{P\text{-}T}(\text{Score}) \approx 0.185\ (65) + 0.63\ (85) + 0.185\ (99) = 83.89.$

d. Five-point approximation using bracket medians:



$E_{BM}(\text{Score}) \approx 0.2\ (68) + 0.2\ (80) + 0.2\ (85) + 0.2\ (93) + 0.2\ (98) = 84.8.$

8.16. We continue with the example given above in 8.12, using the Dow Jones Industrial Average. The CDF is:



a. $E_{P\text{-}T}$(DJIA) = 0.185 (2000) + 0.63 (2600) + 0.185 (3000) = 2563

b. E_{BM}(DJIA) = 0.2 (2100) + 0.2 (2400) + 0.2 (2600) + 0.2 (2750) + 0.2 (2925)
= 2555

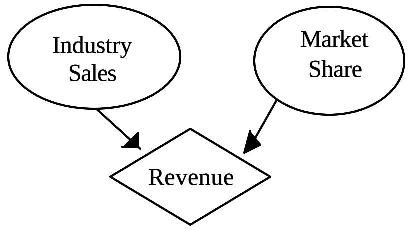
8.17. The issue in this problem is whether an assessment made in one way is better than another. The assessments will probably not be perfectly consistent. That is, P(Mets win series) = *p* will probably not be exactly equal to P(Mets win series | Mets win Pennant) × P(Mets win pennant). For example, P(Mets win series) may be holistically assessed as 0.02, while P(Mets win series | Mets win Pennant) = 0.6 and P(Mets win pennant) = 0.1, giving P(Mets win series) = 0.06.
For many individuals, particularly those with some knowledge of the Mets' current team, the decomposed assessment may be easier, and they may have more confidence in the final result.

8.18. Students must assess P(Hospitalized | Accident), P(Hospitalized | No Accident), and P(Accident). With these assessments, they could calculate

P(Hospitalized) = P(Hospitalized | Accident) P(Accident)
+ P(Hospitalized | No Accident) P(No accident)

It would be possible to decompose the assessment further or in other ways. For example, one might consider the possibility of contracting a serious disease or requiring hospitalization for mental illness.

8.19. This problem is actually a full-scale project requiring considerable research. The students might consider assessing a distribution for industry sales and a distribution for the firm's market share. Together, these two quantities give the distribution of revenue for the firm:



8.20. Tversky and Kahneman (1971) attribute the gambler's fallacy to the representativeness heuristic and a misunderstanding of random processes. People tend to think that small segments of a random process will be highly representative of the overall process. Hence, after a string of red on a roulette wheel, it is thought that black must occur to balance the sample and make it more representative of the overall process (50% red, 50% black). Source: Tversky, A., and Kahneman, D. (1971) "Belief in the Law of Small Numbers," *Psychological Bulletin*, 76, 105-110.

8.21. "Linda" (Problem 7.25) is a classic example of the representativeness heuristic. Many people judge that Linda's description is less representative of a bank teller than of a feminist bank teller.

8.22. The "regression to the mean" phenomenon could easily be at work here. The D is most likely an "outlier" and, hence, is likely to be followed by an improvement.
Another argument is that your parents should not use the D as a basis to compare you to other D students (representativeness heuristic). In contrast, they should consider your "base rate" (as a B student) and not overweight the poor exam performance.

8.23. In principle, the notion of a requisite model is appropriate in answering this question, but the application of the concept is delicate. It is possible to perform sensitivity analysis on, say, a three-point discrete approximation by wiggling the representative points. Do small changes in the 0.05, 0.5, or 0.95 fractiles affect the choice? If not, then further assessments are probably not necessary. If they do, then more assessments, possibly decomposed assessments, and a clearer picture of the CDF are needed to obtain an unequivocal decision.

8.24. There are a variety of different possibilities here. Perhaps the most straightforward is to obtain "upper" and "lower" probabilities and perform a sensitivity analysis. That is, solve the decision tree or influence diagram with each probability. Does the optimal choice change? If not, there is really no problem. If the decision is sensitive to the range of probabilities, it may be necessary to assess the probabilities more carefully.
Another simple solution is to assess a continuous distribution for the probability in question (a "second-order" probability distribution). Now estimate the expected value of this distribution using bracket medians or the Pearson-Tukey method. Finally, use the expected value as the probability in the decision problem.

8.25. a. For each project, the investor appears to believe that

$$
\begin{aligned}
E(\text{profit}) &= 0.5\ (150{,}000) + 0.5\ (-100{,}000) \\
&= 75{,}000 - 50{,}000 \\
&= 25{,}000
\end{aligned}
$$

b. However, since only one of the projects will succeed, he will gain $150,000 for the successful project, but lose $100,000 for each of the other two. Thus, he is guaranteed to lose $50,000 no matter what happens.

c. For a set of mutually exclusive and collectively exhaustive outcomes, he appears to have assessed probabilities that add up to 1.5.

d. "Knowing nothing" does not necessarily imply a probability of 0.5. In this case, "knowing nothing" is really not appropriate, because the investor does know something: Only one project will succeed. If, on top of that, he wants to invoke equally likely outcomes, then he should use P(Success) = 1/3 for each project.

Note that it is also possible to work parts a and b in terms of final wealth. Assume that he starts with $300,000. For part a, expected final wealth, given that he invests in one project, would be:

$$\begin{aligned} E(\text{wealth}) &= 0.5\,(450{,}000) + 0.5\,(200{,}000) \\ &= 325{,}000. \end{aligned}$$

Because $325,000 is an improvement on the initial wealth, the project looks good. For part b, though, if he starts with $300,000 and invests in all three projects, he will end up with only $250,000 for the project that succeeds. As before, he is guaranteed to lose $50,000.

8.26. a. If he will accept Bet 1, then it must have non-negative expected value:

$$P(\text{Cubs win})\,(\$20) + [1 - P(\text{Cubs win})]\,(-\$30) \geq 0.$$

Increasing the "win" amount to something more than $20, or decreasing the amount he must pay if he loses ($30) will increase the EMV of the bet. However, reducing the "win" amount or increasing the "lose" amount may result in a negative EMV, in which case he would not bet. The same argument holds true for Bet 2.

b. Because he is willing to accept Bet 1, we know that

$$P(\text{Cubs win})\,(\$20) + [1 - P(\text{Cubs win})]\,(-\$30) \geq 0,$$

which can be reduced algebraically to

$$P(\text{Cubs win}) \geq 0.60.$$

Likewise, for Bet 2, we know that

$$P(\text{Cubs win})\,(-20) + [1 - P(\text{Cubs win})]\,(\$40) \geq 0.$$

This can be reduced to $P(\text{Cubs win}) \leq 0.67$. Thus, we have $0.60 \leq P(\text{Cubs win}) \leq 0.67$.

c. Set up a pair of bets using the strategy from Chapter 8. From Bet 1 we infer that P(Cubs win) = 0.60, and from Bet 2 P(Yankees win) = 0.33. Use these to make up Bets A and B:

A: He wins 0.4 X if Cubs win
He loses 0.6 X if Yankees win

B: He wins 0.67 Y if the Yankees win
He loses 0.33 Y if the Cubs win

We can easily verify that the EMV of each bet is equal to 0:

$$\begin{aligned} EMV(A) &= P(\text{Cubs win})\,(0.4\,X) + [1 - P(\text{Cubs win})]\,(-\,0.6\,X) \\ &= 0.6\,(0.4\,X) - 0.4\,(0.6\,X) \\ &= 0 \end{aligned}$$

$$\begin{aligned} EMV(B) &= P(\text{Yankees win})\,(0.67\,Y) + [1 - P(\text{Yankees win})]\,(-0.33\,Y) \\ &= 0.33\,(0.67\,Y) - 0.67\,(0.33\,Y) \\ &= 0 \end{aligned}$$

If the Cubs win, his position is:

$$0.4 X - 0.33 Y = W$$

If the Yankees win:

$$-0.6 X + 0.67 Y = Z$$

Following the strategy in the book, set W = Z = -\$100 to be sure that he pays us \$100 net, regardless of what happens:

$$0.4 X - 0.33 Y = -\$100$$
$$-0.6 X + 0.67 Y = -\$100$$

Now solve these two equations for X and Y to obtain X = Y = -\$1500. Thus, the original bets A and B become:

A: He wins -\$600 if Cubs win
He loses -\$900 if Yankees win

B: He wins -\$1000 if the Yankees win
He loses -\$500 if the Cubs win

The minus sign means that he is taking the “other side” of the bet, though (i.e. winning -\$600 is the same as losing \$600). Thus, these two bets really are:

A: He loses \$600 if Cubs win
He wins \$900 if Yankees win

B: He loses \$1000 if the Yankees win
He wins \$500 if the Cubs win

Finally, compare these bets to Bets 1 and 2 in the book. He said he would bet on the Cubs at odds of 3:2 or better, but we have him betting on the Cubs (in Bet B) at odds of 2:1, which is worse. (That is, he has to put up 2 dollars to win 1, rather than 1.5 to win 1.) The same problem exists with bet A: he is betting on the Yankees at odds of 2:3, which is worse than 1:2. As a result, he will not accept either of these bets!
The reason for this result is that the solutions for X and Y are negative. In fact, it is possible to show algebraically that if A and B are both negative, then X and Y will both be negative. This has the effect of reversing the bets in such a way that your friend will accept neither. The conclusion is that, even though his probabilities appear incoherent, you cannot set up a Dutch book against him.

8.27. a, b. Most people choose A and D because these two are the options for which the probability of winning is known.

c. Choosing A and D may appear to be consistent because both of these involve known probabilities. However, consider the EMVs for the lotteries and the implied values for P(Blue). If A is preferred to B, then

$$\text{EMV(A)} > \text{EMV(B)}$$
$$\frac{1}{3}(1000) > \text{P(Blue)}\,(1000)$$
$$\text{P(Blue)} < \frac{1}{3}.$$

However, if D is preferred to C, then

$$EMV(D) > EMV(C)$$

$$P(Blue)\,(1000) + P(Yellow)\,(1000) > \frac{1}{3}(1000) + P(Yellow)\,(1000)$$

$$P(Blue) > \frac{1}{3}.$$

The inconsistency arises because it clearly is not possible to have both $P(Blue) < \frac{1}{3}$ and $P(Blue) > \frac{1}{3}$.

(Exactly the same result obtains if we use the utility of $1000, instead of the dollar value.)

Case Study: Assessing Cancer Risk — From Mouse to Man

1. Some assumptions and judgments that must be made:

- Organisms react in a specified way to both high and low doses of the substance. Researchers have developed dosage response models, but validating those models has been difficult. Threshold effects may exist; at a low dosage level, an organism may be able to process a particular toxin effectively, although at a higher level (beyond the threshold) reactions to the toxin may appear.
- The test species is judged to react to the toxin in the same way that humans do. Some evidence indicates, though, that human livers are better at processing toxins than mouse or rat livers. Thus, dosage responses may not be the same across species.
- Lab and field exposures are similar in nature. However, in the field many more complicating and possibly interactive effects exist.

The first two assumptions take shape in what is called a "dose-response" model. This is a mathematical relationship that estimates the magnitude of the physiological response to different doses of the chemical.

What kinds of evidence would help to nail down the effects of toxic substances? Long term studies of toxins in the field and in the lab would be most useful. We need to know effects of low doses on humans in the field, in order to refine the human dose-response model, but this information may be very difficult to gather. Certainly, no controlled studies could be performed!

2. The question is whether one bans substances on the grounds that they have not been demonstrated to be safe, or does one permit their use on the grounds that they have not been demonstrated to be dangerous. The choice depends on how the decision maker values the potential economic benefits relative to the potential (but unknown) risks.

3. The issue of credibility of information sources is one with which scientists are beginning to wrestle, and it is a complicated one. Intuitively, one would give more weight to those information sources that are more credible. However, systematic ways of assessing credibility are not yet available. Furthermore, the overall impact of differences in source credibility on the decision maker's posterior beliefs is unclear.

Case Study: Breast Implants

1. There clearly are differences in the quality of information that is available in most situations. Science teaches us to beware of inferences based on small samples, yet anecdotes can be used to paint compelling scenarios. Are judges prepared to make judgments regarding the quality of information presented as "scientific"? How can a judge, not trained in the science himself, be expected to make reasonable judgments in this respect? And if a judge is ill prepared, what about jurors?

2. The questions asked in the last paragraph of the quoted passage clearly relate primarily to preferences. In a democratic, capitalistic society, we generally assume that individual consumers should get to make their own decisions, based on their own preferences. In this case, however, the issue of preference deals with how much risk is acceptable. And that question presumes that the decision maker knows what the risk is.

The level of risk, however, is a matter of uncertainty ("facts," in contrast to "values"), and it takes experts to measure that risk. In situations where individual consumers cannot realistically be expected to understand fully the risks, we often expect the government to step in to regulate the consumer risk. It is not so much a matter of protecting the consumer from himself as it is being sure that the risks are appropriately measured, the information disseminated to the consumers, and , where appropriate, appropriate standards set.

Case Study: The Space Shuttle *Challenger*

1. With little or no information, does one refrain from launching the spacecraft on the grounds that no proof exists that the launch would be safe, or does one launch on the grounds that there is no proof that doing so is unsafe. Since the *Challenger* accident, NASA has implemented a system whereby the policy is clearly stated: Do not launch if there are doubts as to the safety.

2. These subjective estimates made by different people are based on different information and different perspectives, and are used for different purposes. It is important for a decision maker to look beyond the biases, try to judge the "credibility" of the judgments, and take these into account in developing his or her own probabilities or beliefs. The same caveats as in question 3 of the Cancer Risk case apply here, however. That is, even though it seems appropriate to weight more credible sources more heavily, neither precise methods nor an understanding of the impact of doing so exist at this time.

3. The overall effect of slight optimism in making each individual assessment would be a very overoptimistic probability of failure. That is, P(failure) would wind up being much lower than it should be.

4. Reichhardt's editorial raises the question of what is an "acceptable risk." How should society determine what an acceptable risk would be? How society should choose an acceptable risk level for enterprises such as nuclear power generation, genetic research, and so on, has been a hotly debated topic. Furthermore, different people are willing to accept different levels of risk. For example, an astronaut, thrilled with the prospect of actually being in space, may be more willing to accept a high level of risk than a NASA administrator who may be subject to social and political repercussions in the event of an accident. Because of the diversity of preferences, there is no obvious way to determine a single level of acceptable risk that would be satisfactory for everyone.

Further reading: Fischhoff, B., S. Lichtenstein, P. Slovic, S. Derby, & R. Keeney (1981) *Acceptable Risk*. Cambridge: Cambridge University Press.