LECTURE NOTES: EMGT 234

MODELING EVALUATION OF CONSEQUENCES

SOURCE:

Chapter 13, 14
Making Hard Decisions
R.T. Clemen

CONTENTS

1. RULES OF CLEAR THINKING

- Relate to the consistency with which an individual expresses preferences amongst a series of risk prospects

AXIOMS = RULES OF CLEAR THINKING

1. Ordering and Transitivity

$$\begin{cases} \prec & \text{Indicates preference of right hand side to left hand side} \\ \succ & \text{Indicates preference of left hand side to right hand side} \\ \sim & \text{Indicates indifference between right hand side and left hand side} \end{cases}$$

Suppose Alternatives A_1, A_2, A_3 are available:

ORDERING:

- $A_1 \succ A_2$
- $A_1 \prec A_2$
- $A_1 \sim A_2$

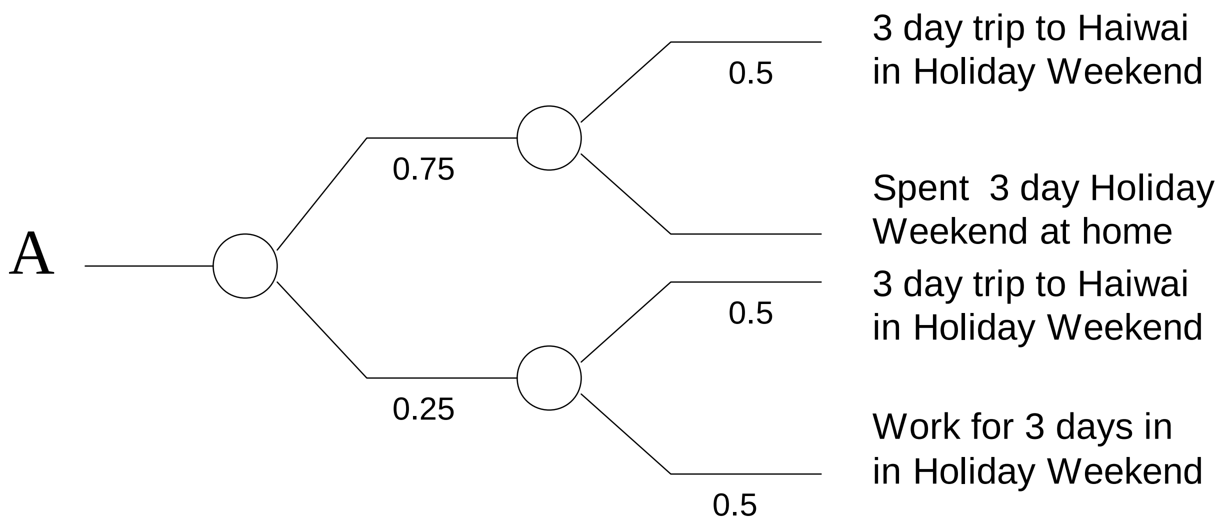
TRANSITIVITY:

$$A_1 \succ A_2, A_2 \succ A_3 \Rightarrow A_1 \succ A_3$$

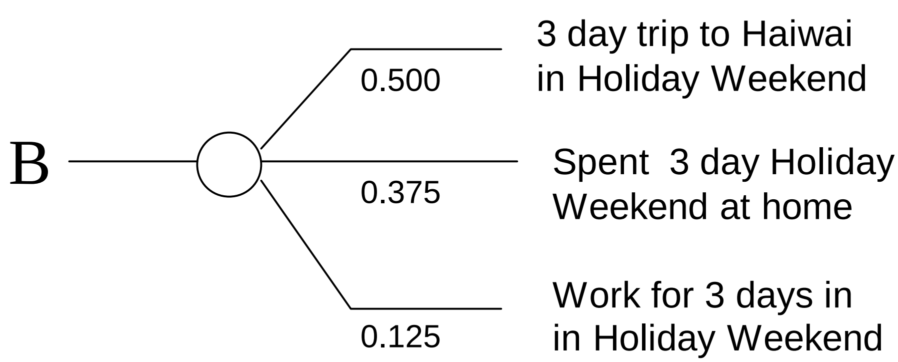
2. Reduction of Compound Uncertain Events

A Decision Maker (D.M.) is indifferent between a compound uncertain event as determined by reduction using standard probability manipulations.

Alternative A:



Alternative B:



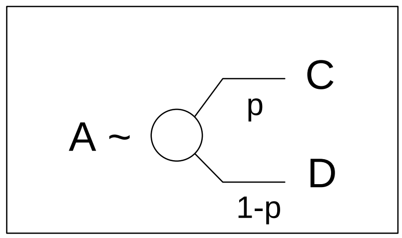
Conclusion: DM is indifferent between alternative A and B

$$A \sim B$$

3. Continuity

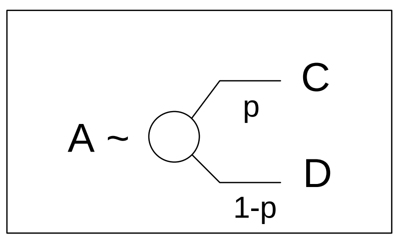
If $C \prec A \prec D$ you can always design an uncertain event such that the DM is indifferent between outcome A and the uncertain event involving $C \,\&\, D$.

If $C \prec A \prec D$, there is a probability **p> 0**, such that

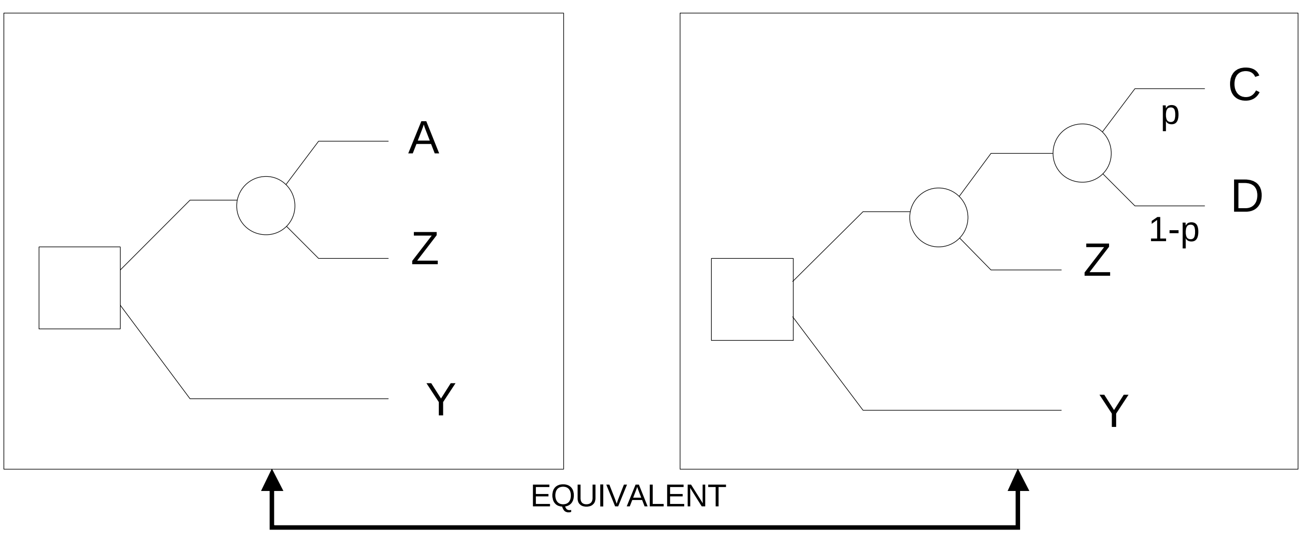


4. Substitutability

If

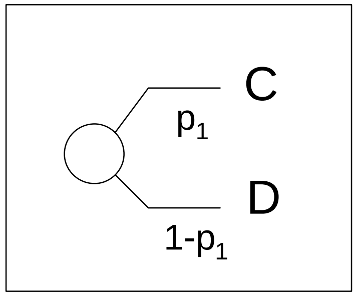


Then

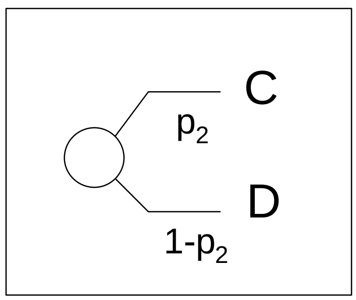


5. Monotonicity

Suppose $C \succ D$ and $p_1 > p_2$, then



is preferred ("$\succ$") over



6. Invariance

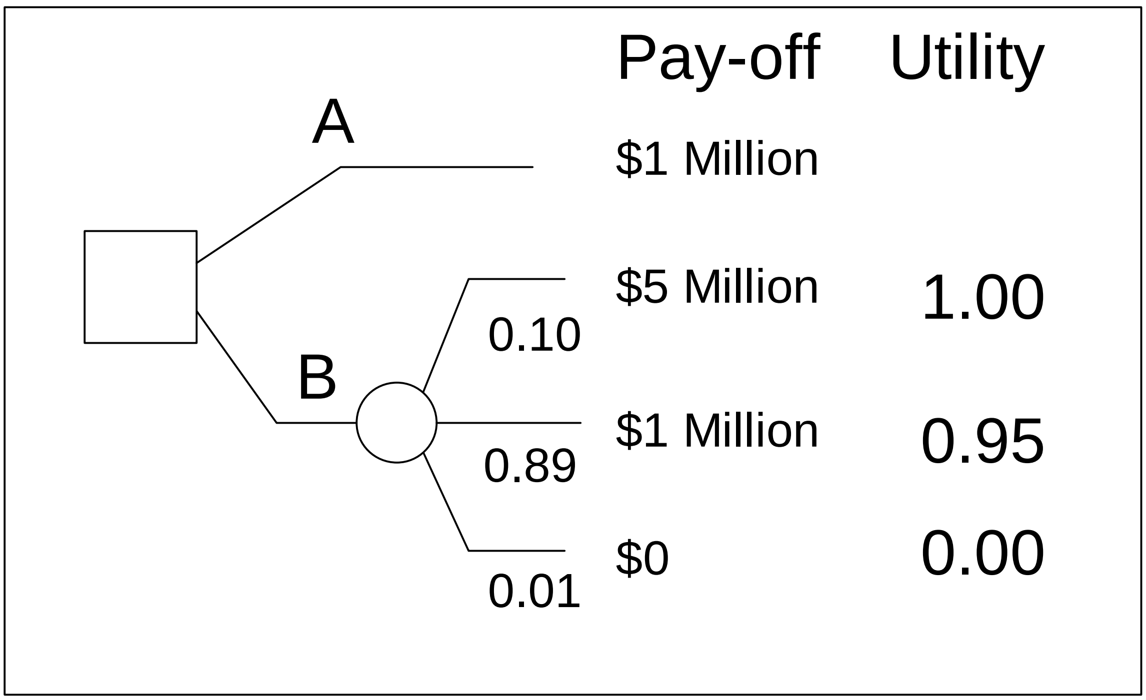
All you need to determine a DM preference among alternatives are the uncertain events associated with the alternatives, their probabilities and the payoff (or consequences) associated with the outcomes of the uncertain events.

7. Finiteness

No consequence is considered INFINITLY BAD or INFINITLY WORSE. In other words, you can always imagine something "better" or something "worse" than the consequences present in the decision problem.

Theorem:
Given a set of alternatives, with associated uncertain events, probabilities and consequences.

If axioms 1 to 7 hold, one can **always** attach real numbers to the consequences (=utilities) such that the preference structure imbedded by **making decisions based on maximimzing expected utility** coincides with ones internal preference structure.



$$EU(B) = \sum_{i=1}^{3} p_i * u_i = 0.10 * 1.00 + 0.89 * 0.95 + 0.01 * 0.00 = 0.9455$$

Which alternative is preferred?

Suppose in above example u_i's are transformed into

$$a * u_i + b$$

where **a** and **b** constants and **a > 0.**

- $EU(B) = \sum_{i=1}^{3} p_i * (a * u_i + b) =$

$$\sum_{i=1}^{3} p_i * a * u_i + \sum_{i=1}^{3} p_i * b =$$

$$a * \sum_{i=1}^{3} p_i * u_i + b * \sum_{i=1}^{3} p_i = a * 0.9455 + b$$

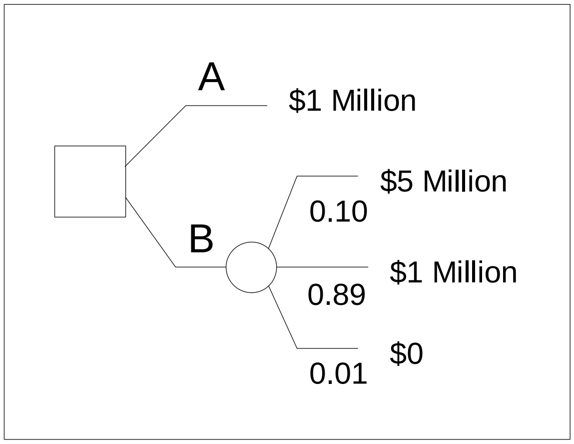
- $EU(A) = a * 0.95 + b$

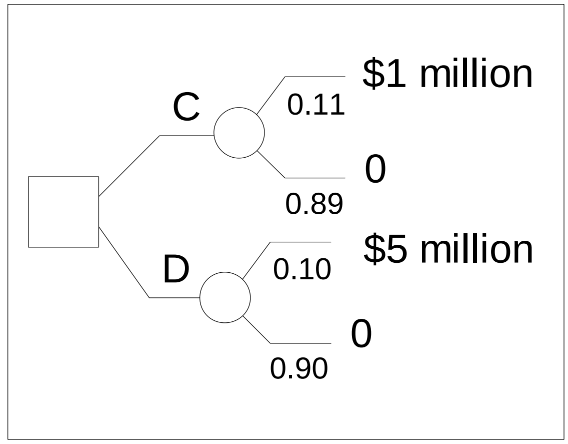
Which one is preferred using transformed u_i's?

Conclusion:

- No matter what the values of **a** and **b** are , our preference structure remains the same as long as **a** is positive.
- Utilities are used to rank consequences and alternative by using expected utilities.

ALIAS PARADOX





Which one would you prefer?

		TICKETS		
	OPTIONS	**1**	**2 to 11**	**12 to 100**
BET 1	**A**	$1 million	$1 million	$1 million
	B	$0	$5 million	$1 million
BET 2	**C**	$1 million	$1 million	$0
	D	$0	$5 million	$0

Note:
To interpret differences between expected utilities as a measure for **the degree of preference,** additional axioms are required.

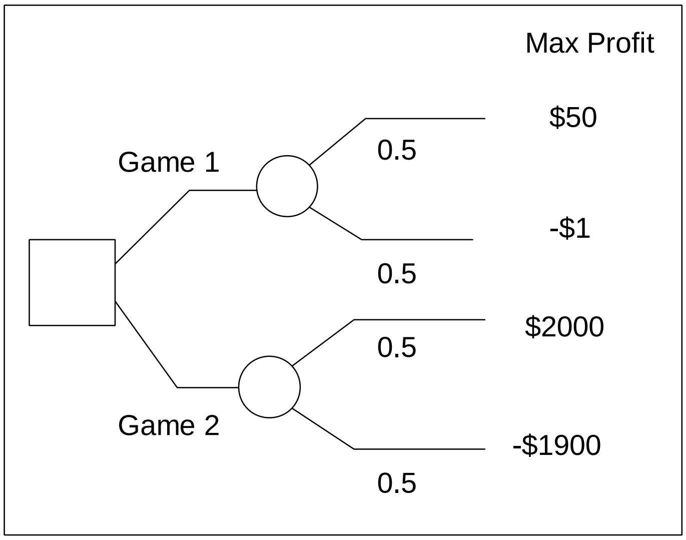
If additional axioms apply to a particular utility function, the utility function may be interpreted as a value function.

Ralph Keeney (1994), "Value Focussed Thinking"

INDIVIDUAL UTILITY FUNCTIONS

Making decisions based on EMV
(=Expected Monetary Average) is convenient
but may lead to counter intuitive decisions.

Example:



Which Game Would You Prefer?
Most people prefer Game 1 over Game 2.

But:

GAME 1

Prob	Payoff	Prob*Payoff
0.5	$30.00	$15.00
0.5	-$1.00	-$0.50
	EMV=	$14.50

GAME 2

Prob	Payoff	Prob*Payoff
0.5	$2,000.00	$1,000.00
0.5	-$1,900.00	-$950.00
	EMV=	$50.00

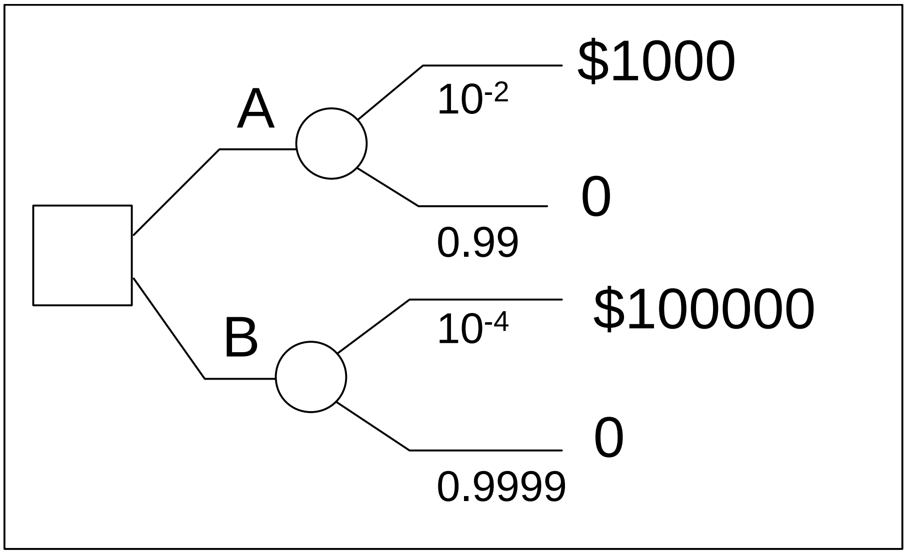
Why?

- Expected Monetary Value =(EMV) is a long term average, whereas the games are only played once.
- EMV does not take into account the risk (=fear) involved with loosing larger amounts of money.

Solution?

Capture RISK ATTITUDES in attribute ($ in above problem) by modeling individual utility function

Example:



- Somebody who is **Risk Neutral** is indifferent between A and B. Makes decisions based on EMV
- Somebody who is **Risk Averse** prefers A
- Somebody who is **Risk Seeking** prefers B

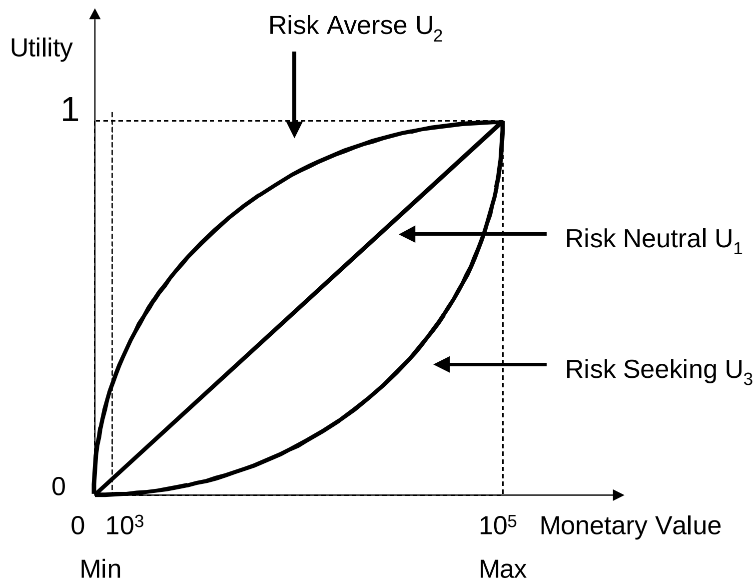
Capture behavior above through use of

Individual Utility Functions =
A method of modeling risk attitude by transforming $ into Utility Units (=Utils)

Utility Function: U(X) (in Utils)

in Dollars

Modeling Risk Attitudes



Note: Utility of Best Case = 1 , Utility of Worst Case = 0.

Risk Neutral:

$$U_1(10^3) = \frac{U_1(10^5)}{10^2} \Rightarrow \begin{cases} EU_1(A) = 10^{-2} \cdot U_1(10^3) = 10^{-2} \dfrac{U_1(10^5)}{10^2} \\ EU_1(B) = 10^{-4} \cdot U_1(10^5) \end{cases} \Rightarrow A \sim B$$

Risk Averse:

$$U_2(10^3) > \frac{U_2(10^5)}{10^2} \Rightarrow \begin{cases} EU_2(A) = 10^{-2} \cdot U_2(10^3) > 10^{-2} \dfrac{U_2(10^5)}{10^2} \\ EU_2(B) = 10^{-4} \cdot U_2(10^5) \end{cases} \Rightarrow A \succ B$$

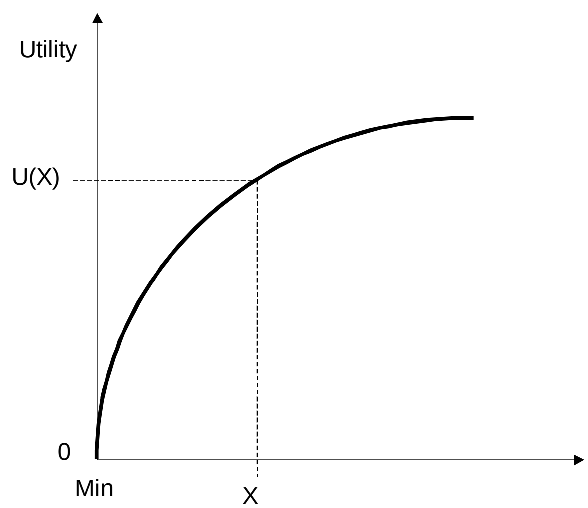
Risk Seeking:

$$U_3(10^3) < \frac{U_3(10^5)}{10^2} \Rightarrow \begin{cases} EU_3(A) = 10^{-2} \cdot U_3(10^3) < 10^{-2} \dfrac{U_3(10^5)}{10^2} \\ EU_3(B) = 10^{-4} \cdot U_3(10^5) \end{cases} \Rightarrow A \prec B$$

Note:

- All Utility Functions are upward sloping indicating more wealth is better
- Curvature (=Concave, Linear, Convex) indicates Risk Attitude.
- Utility Function may be specified in different formats

1. Graphical Format:



2. Tabular Format:

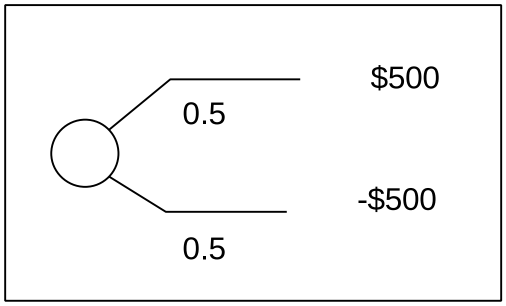
Wealth	Utility
0	0.15
400	0.47
600	0.65
1000	0.93
1500	1.24
2500	1.50

3. Functional Format:

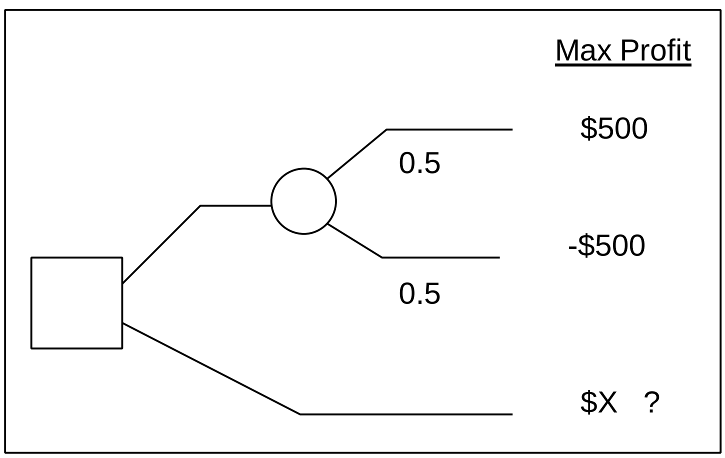
$$U(x) = Log(x), U(x) = 1 - \exp(-\frac{x}{R}), U(x) = \sqrt{x}$$

Example:

You own the following bet or game:



You friend approaches you and asks whether you would like to trade the game. You are faced with the following decision problem.



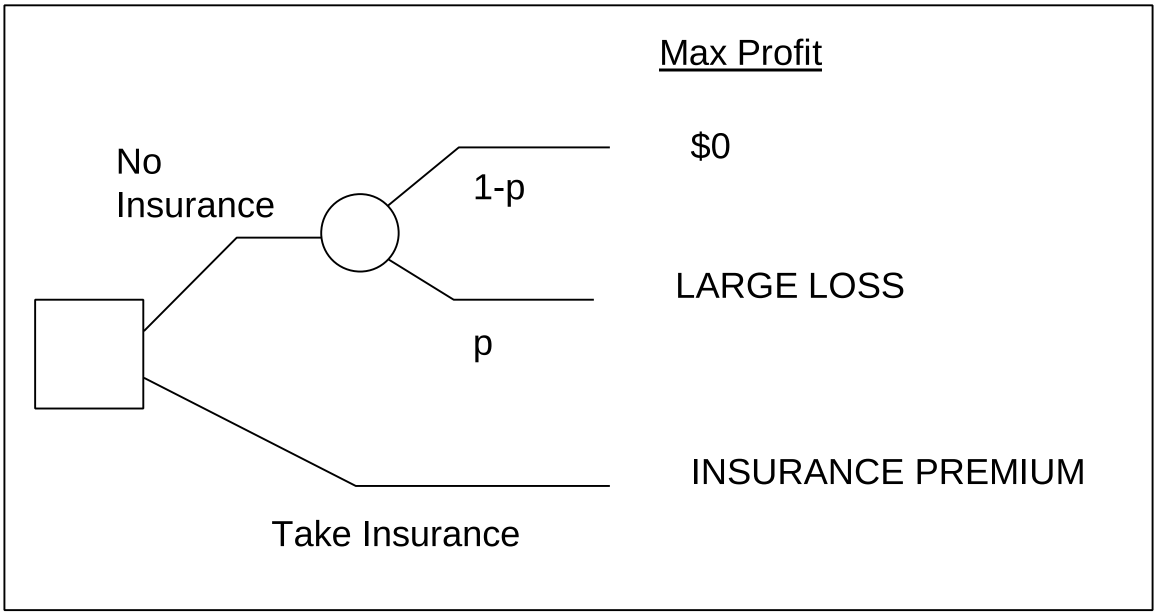
How Much Should You Charge or
Should You Give It Away For Free or
Would You Pay Your Friend ?

Answer depends on your Risk Attitude:

The EMV of your game is $0

- You are Risk Neutral: You give the bet away for free
- You are Risk Seeking: You charge your friend an amount.
- You are Risk Averse: You are willing to give the bet away and pay your friend to accept the bet.

Can We Think Of Such An Example In Real Life?



Insurance charges more than the EMV.
If you pay for insurance you are risk averse.

Question:

Can you imagine a premium at which you would decide not to take insurance given you had the choice?

Utility Function Assessment

- Risk Attitude is personal ⇒ Use of Subjective Judgment

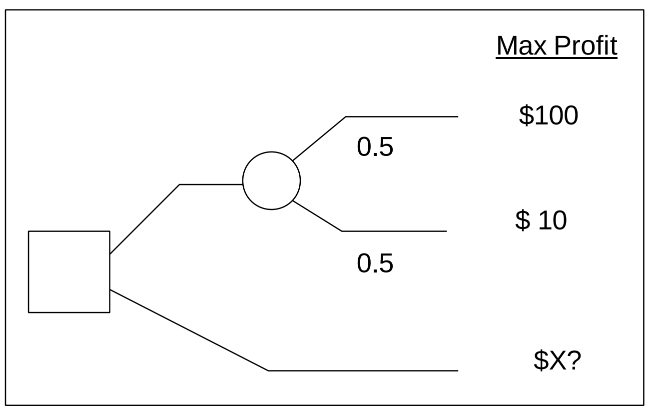
STEP 1: Set U(Min) = 0, U(Max) = 1

Example:
Suppose you are encountering an investment decision which a payoff that ranges from \$10 to \$100.

$$U(\$10)=0,\ U(\$100)=1$$

STEP 2: Asses utility for several intermediate values using reference lotteries that ask for Certainty Equivalents

Example:

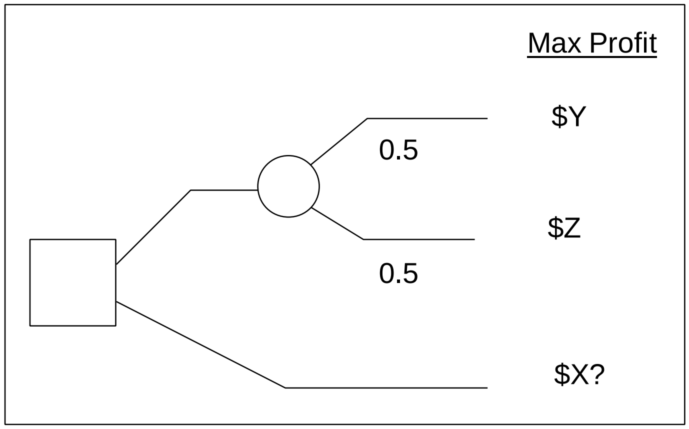


For how much money are you willing to trade?

Answer: \$30 ⇒ $U(\$30) = 0.5*U(\$10) + 0.5*U(\$100)$
$= 0.5*0 + 0.5*1 = 0.5$

In general:

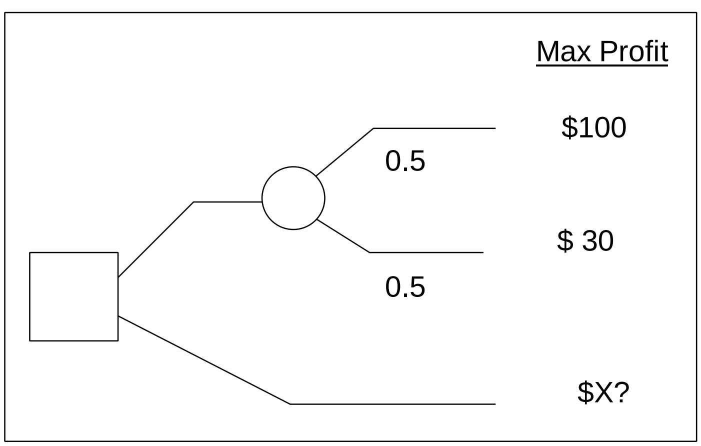
Suppose you know U(Y), U(Z). Ask for the CE using the following reference lottery.



Then: $U(X) = 0.5*U(Y) + 0.5*U(Z)$

Example: Suppose you want to know the utility of an amount between \$30 and \$100.

Note: U(\$100) = 1, U(\$30)=0.5



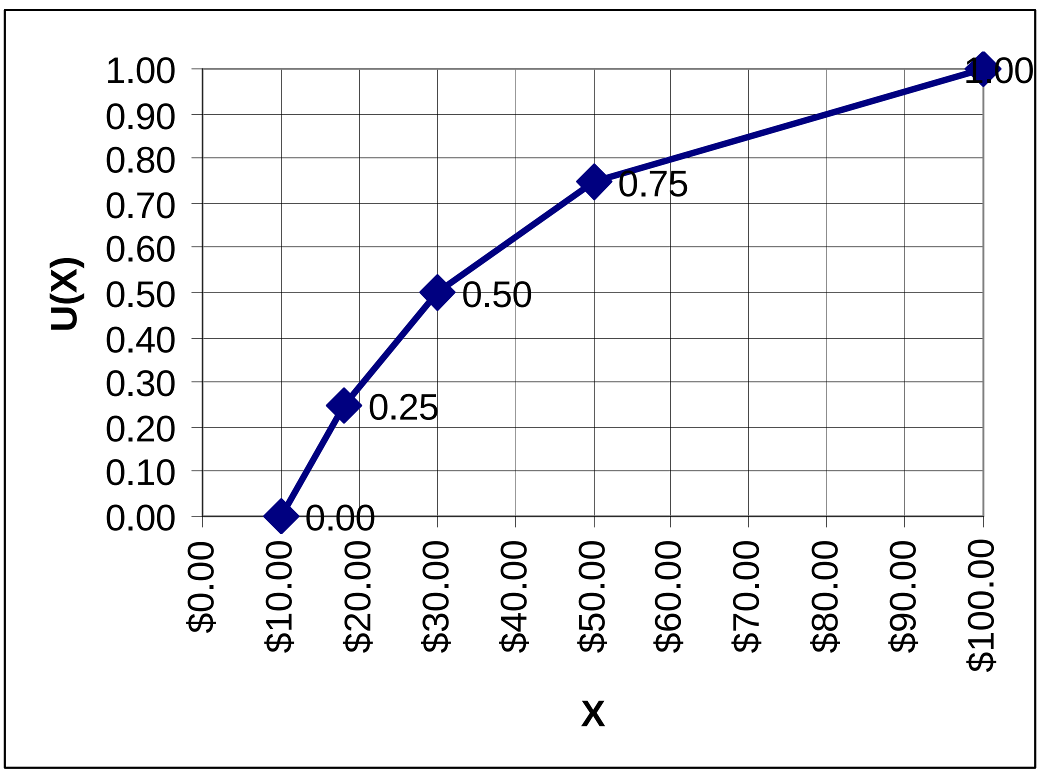
For how much money are you willing to trade?

Answer: \$50 $\Rightarrow$ U(\$50) $= 0.5*U(\$30) + 0.5*U(\$100)$
$= 0.5*0.5 + 0.5*1 = 0.75$

STEP 3:
Approximate Utility Function using Straight Line Approximation.

Example:

X	U(X)
\$10.00	0.00
\$18.00	0.25
\$30.00	0.50
\$50.00	0.75
\$100.00	1.00



Modeling Utility Functions discussed so far:

- Captures Risk Attitudes towards Monetary-Payoff

Utility Functions can be applied to model risk attitude with respect to other attributes as well e.g.:

- Market Share (in %)
- Death Toll in Transportation Accidents (in # Deaths)
- Etc.

1. If you feel **one unit** of the attribute of a fundamental objective is always worth **the same** to you in terms of monetary dollars, you can establish such a utility function by:

STEP 1: Establish Utility Function for Monetary-Payoff

STEP 2: Establish Monetary Equivalent of one unit on the measurement scale of the fundamental objective.

STEP 3:Transform Utility Function for Monetary-Payoff to Utility Function for other fundamental objective.

2. If you feel the above does not apply, you need to asses the utility function directly in terms of the attribute of the fundamental objective.

Example: Monetary Value Life-Saving

Life Valuation for Purposes of Cost-Benefit Analyses
(Source: Henley Kumamoto. 1981)

Approaches	Typical Values	Some Limitations
(1) Implicit Value	\$9,000-\$9,000,000	Assumes past decisions are optimal
(2) Human Capital	\$100,000-\$400,000	Based solely on life-time income. Ignores individual Preferences. Discriminates against unproductive members of society
(3) Insurance Premiums	Wide Range	Does not take into account individuals's interest in protecting his own life
(4) Court Awards	\$250,000	Based on lost income
(5) Willingness to Pay	\$180,000-\$1,000,000	Difficult to estimate. Depends on Risk Situation

Summary: All measures depend to some extent on the lifetime earning potential of the individuals at risk and ignore perception of seriousness.

Conclusion: Cannot be rigorously determined. Choose value (say $300,000) according to personal values, third party interests and psychological factors.

STEP 1:
Utility for Monetary-Payoff: $U(X) = a*Ln(X) + b, a>0$

STEP 2:

Y : Measured in Lives, X : Measured in Dollars
"1 Life saved is equivalent to (say) $300,000"

$$X=300000*Y$$

STEP 3:

$$U(X) = U(300000*Y)= a*Ln(300000*Y) + b$$
$$= a*Ln(Y) + Ln(300000) + b$$

Conclusion:
Preference Structure towards “Lifes Saved” and “Monetary Gains” is the same, due to assumptions of logarithmic utility and constant substitution of 1 life in term of monetary equivalent.

3. MULTI ATTRIBUTE UTILITY FUNCTIONS

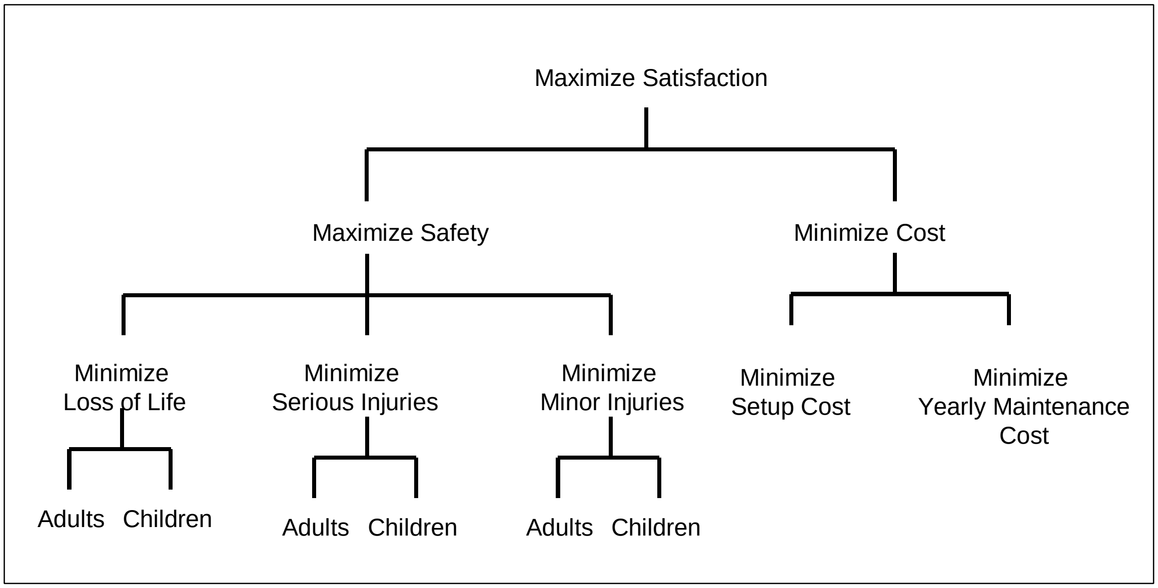
What if more than risk attitude and monetary pay-off in the potential outcome is important?

A doctor prescribing medical treatment must consider a variety of issues:

- Potential Health complications for the patient (perhaps death)
- Money cost to the patients
- Patient's time spent being treated
- Cost to insurance companies
- Payments to doctor.
- Utilization of resources (nurses, hospital space, equipment)
- Information gained in treating this patient (may be helpful in treating others.

Identifying objectives is Creative Process

- **STEP 1:** Establish **Fundamental Objective Hierarchy**



- **STEP 2:** Classify how to **measure (operationalize)** Fundamental Objectives

OBJECTIVE	ATTRIBUTE
Maximize profit	Money (for example dollars)
Maximize Revenue	Money (for example dollars)
Maximize Savings	Money (for example dollars)
Minimize Cost	Money (for example dollars)
Maximize Market Share	Percentage
Maximize Rate of Return	Percentage
Maximize proximity	Miles, minutes
Maximize Safety	# Deaths

When is an attribute operational?

1. Can you explain to someone what to measure and why?
2. Does it take a reasonable amount of effort to measure?
3. If the measurement were given to you by someone else, could you tell how well the objective was achieved. (Usually this means identifiying a worst and best case).

- **STEP 3:** Establish an individual utility function for each attribute associated with a fundamental objective.

- **STEP 4:** Establish a MULTI ATTRIBUTE UTILITY FUNCTION by combining the individual utility functions.

Assumption of Additive Independence
$\Rightarrow$ Additive Utility Function

In words:

No matter how well or bad other objectives are achieved, I will value the achievement of a particular objective the same

ADDITIVE MULTI-ATTRIBUTE UTILITY FUNCTION

STEP 1: Establish a range for each objective i: (A_i, B_i)

STEP 2: Establish a utility function for each objective i: $U_i\ (X_i)$

STEP 3: Establish an importance weight w_i for each objective.

STEP 4: Calculate the combined utility for all objectives

$$U(X_1, \cdots, X_n) = \sum_{i=1}^{n} w_i * U_i(X_i)\ , \sum_{i=1}^{n} w_i = 1, w_i > 0$$

ASSESSING WEIGHTS USING ANALYTICAL HIERARCHY PROCESS

EXAMPLE: SELECTING A JOB

When multiple objectives are important to a Decision Maker, it is often difficult to choose between alternatives.

Example:
You are choosing a job. One might offer the highest starting salary, but rate poorly on your other objectives: quality of life, closeness to your family. Another job offer might rate highly on these latter objectives but has a relatively low starting salary. Which one do you choose?

THOMAS SAATY's ANALYTICAL HIERARCHY PROCESS

Provides a powerfool tool that can be used to make decisions where multiple objectives are present.

Four Objectives:
Objective 1: High Starting Salary
Objective 2: Quality of life in city where job is located
Objective 3: Interest of Work
Objective 4: Nearness of job to family.

PAIRWISE COMPARISON MATRIX OF OBJECTIVES

Suppose we have N objectives. Then consider the N*N Matrix A, such that:

$$A = \begin{pmatrix} a_{1,1} & \cdots & a_{1,N} \\ \cdots & \ddots & \vdots \\ a_{N,1} & \cdots & a_{N,N} \end{pmatrix}$$

where

$a_{i,j}$= "How much more important is objective i compared to to Objective j.

$$a_{i,j} = \begin{cases} >1 & \text{Objective i is more important than Objective j} \\ 1 & \text{Objective i is as Important as Objective j} \\ <1 & \text{Objective i is less Important as Objective j} \end{cases}$$

ASSUMPTION:

- There exists a set of numbers w_i , i=1,…, n such that:

w_i = Relative importance of objective i compared to the other objectives.

$$\sum_{i=1}^{n} w_i = 1, w_i > 0, a_{i,j} = \frac{w_i}{w_j}$$

Note that:

$$a_{j,i} = \frac{w_j}{w_i} = \frac{1}{\frac{w_i}{w_j}} = \frac{1}{a_{i,j}}$$

BUT HOW DO WE GET SPECIFIC VALUES FOR $a_{i,j}$?

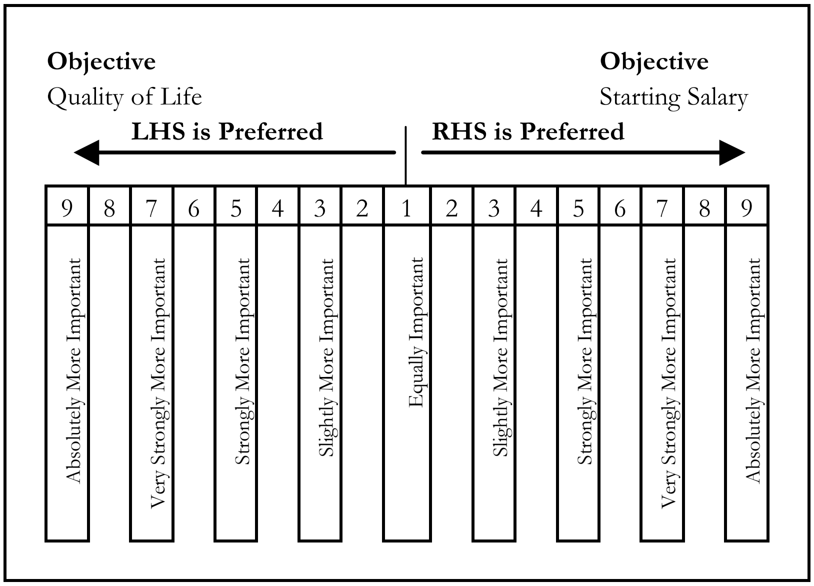
STEP 1:
Introduce a quantitative scale for measuring importance

1: Equally Important
3: Slightly More Important
5: Strongly More Important
7: Very Strongly More Important
9: Absolutely More Important

CRITIQUE: Where does this scale come from? Why five categories and why a scale from 1 to 9? Do the analysis result depend one these choices?

ANSWER: There is instability in the final scores that are being calculated and in the **RANKINGS** of the alternatives as well (AHP is not perfect, but very practical and widely applied method).

STEP 2:
Develop a questionnaire. Use an attractive graphical format for the questions. For example:



Total number of objectives is N. What is the number of pairwise comparison you need to ask?

ANSWER: $\binom{N}{2}$

Answers are typically summarized in a Matrix Form

Back to our Job Selection Example:

$$A = \begin{pmatrix} 1 & 5 & 2 & 4 \\ \frac{1}{5} & 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 2 & 1 & 2 \\ \frac{1}{4} & 2 & \frac{1}{2} & 1 \end{pmatrix}$$

HOW DO WE CALCULATE THE WEIGHT VECTOR

$$\underline{w} = (w_1, w_2, \cdots, w_n)^T$$

FROM THIS MATRIX?

Consider the case of 4 Objectives:

$$A = \begin{pmatrix} \frac{w_1}{w_1} & \frac{w_1}{w_2} & \frac{w_1}{w_3} & \frac{w_1}{w_4} \\ \frac{w_2}{w_1} & \frac{w_2}{w_2} & \frac{w_2}{w_3} & \frac{w_2}{w_4} \\ \frac{w_3}{w_1} & \frac{w_3}{w_2} & \frac{w_3}{w_3} & \frac{w_3}{w_4} \\ \frac{w_4}{w_1} & \frac{w_4}{w_2} & \frac{w_4}{w_3} & \frac{w_4}{w_4} \end{pmatrix}$$

STEP 3. Calculate:

$$\sum_{i=1}^{N} a_{i,j} = \sum_{i=1}^{N} \frac{w_i}{w_j} = \frac{\sum_{i=1}^{N} w_i}{w_j} = \frac{1}{w_j}$$

STEP 4. Calculate Normalized matrix A_{Norm} with elements

$$\tilde{a}_{i,j} = \frac{a_{i,j}}{\sum_{i=1}^{N} a_{i,j}} = \frac{a_{i,j}}{\frac{1}{w_j}} = \frac{\frac{w_i}{w_j}}{\frac{1}{w_j}} = w_i$$

$$A_{Norm} = \begin{pmatrix} w_1 & w_1 & w_1 & w_1 \\ w_2 & w_2 & w_2 & w_2 \\ w_3 & w_3 & w_3 & w_3 \\ w_4 & w_4 & w_4 & w_4 \end{pmatrix}$$

If Subjective Judgment is perfect the top
the elements in each row should be the same

ANSWER MATRIX

	Salary	Life quality	Work interest	Near family
Salary	1	5	2	4
Life quality	1/5	1	1/2	1/2
Work interest	1/2	2	1	2
Near family	1/4	2	1/2	1
Sum	**1.95**	**10**	**4**	**7.5**

"NORMALIZED MATRIX"

	Salary	Life quality	Work interest	Near family
Salary	0.513	0.500	0.500	0.533
Life quality	0.103	0.100	0.125	0.067
Work interest	0.256	0.200	0.250	0.267
Near family	0.128	0.200	0.125	0.133
Sum	**1.0000**	**1.0000**	**1.0000**	**1.0000**

CONCLUSION:
Subjective Judgment is NOT PERFECT

STEP 4. Estimate Weight w_i for objective i uch that:

$$w_i = \frac{1}{N} \sum_{j=1}^{N} \tilde{a}_{i,j}$$

Back to our Job Selection Example:

Normalized matrix

	Salary	Life quality	Work interest	Near family	Weights
Salary	0.513	0.500	0.500	0.533	0.5115
Life quality	0.103	0.100	0.125	0.067	0.0986
Work interest	0.256	0.200	0.250	0.267	0.2433
Near family	0.128	0.200	0.125	0.133	0.1466
Sum	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1**

CHECKING CONSISTENCY IN JUDGMENTS

$$A = \begin{pmatrix} \frac{w_1}{w_1} & \frac{w_1}{w_2} & \frac{w_1}{w_3} & \frac{w_1}{w_4} \\ \frac{w_2}{w_1} & \frac{w_2}{w_2} & \frac{w_2}{w_3} & \frac{w_2}{w_4} \\ \frac{w_3}{w_1} & \frac{w_3}{w_2} & \frac{w_3}{w_3} & \frac{w_3}{w_4} \\ \frac{w_4}{w_1} & \frac{w_4}{w_2} & \frac{w_4}{w_3} & \frac{w_4}{w_4} \end{pmatrix} \underline{w} = (w_1, w_2, w_3, w_4)^T$$

HENCE!

$$A\underline{w} = \begin{pmatrix} 4w_1 \\ 4w_2 \\ 4w_3 \\ 4w_4 \end{pmatrix}$$

THEREFOR:

$$\frac{A\underline{w}}{\underline{w}} = \begin{pmatrix} \frac{4w_1}{w_1} \\ \frac{4w_2}{w_2} \\ \frac{4w_3}{w_3} \\ \frac{4w_4}{w_4} \end{pmatrix} = \underline{4}$$

Back to our Job Selection Example:

Product	Ratios
2.0774	4.0611
0.3958	4.0161
0.9894	4.0672
0.5933	4.0459
CI	**0.0159**

Consistency Index:

$$CI = \frac{\text{Average Ratio's - n}}{\text{n - 1}}$$

Note That:
When expert judgment is perfect CI should be zero

- Suppose an Expert would be filling out the questionnaire at random and we would calculate the associated CI, what would that value be?

The experiment above can be conducted using computerized random answer. By conducting this experiment a great number of times one can calculate:
RI – INDEX = average CI – index.

If pairwise comparison matrix is an N*N Matrix the following RI' indices have been calculated:

n	2	3	4	5	6	7	8	9	10
RI	0	0.58	0.9	1.12	1.24	1.32	1.41	1.45	1.51

THOMAS SAATY's suggests:

$$\begin{cases} \dfrac{CI}{RI} < 0.10 & \text{Degree of consistency satisfactory} \\ \dfrac{CI}{RI} > 0.10 & \text{Serious Inconsistencies} \end{cases}$$

Back to our Job Selection Example:

- **CI = 0.0159**
- **RI=0.9**
- **CI/RI=0.0176**

CONCLUSION : JUDGEMENT IS CONSISTENT