Theorem: Let U be a uniform random variable on $[0,1]$. Let X be a continuous random variable with cumulative distribution function (cdf) $F(\,\cdot\,)$. Let Y be defined such that $Y = F^{-1}(U)$. Proof that Y has cdf $F(\,\cdot\,)$.

Proof:

Note that, a random variable U is uniformly distributed on $[0,1]$ if the following holds

$$U \sim Uniform[0,1] \Leftrightarrow Pr[U \leq u] = u. \tag{1}$$

Also note that for any function $g(\,\cdot\,)$ with a unique inverse function $g^{-1}(\,\cdot\,)$, the following holds

$$g\{g^{-1}(x)\} = x \vee g^{-1}\{g(x)\} = x. \tag{2}$$

The function $F(\,\cdot\,)$ is the cdf of a continuos random variable X, hence its inverse $F^{-1}(\,\cdot\,)$ is uniquely defined on $[0,1]$. Let $Y = F^{-1}(U)$ and consider

$$Pr[Y \leq y] = Pr[F^{-1}(U) \leq y]. \tag{3}$$

Since the function $F(\,\cdot\,)$ is a strictly increasing function we have with (3) that

$$Pr[Y \leq y] = Pr[F\{F^{-1}(U)\} \leq F(y)]. \tag{4}$$

Equations (2) and (4) yield

$$Pr[Y \leq y] = Pr[U \leq F(y)] \tag{5}$$

and from (1) and (5) we have $Pr[Y \leq y] = F(y)$. □