Three-Point Lifetime Distribution Elicitation for Maintenance Optimization in a Bayesian Context

"Presentation Short Course: Beyond Beta and Applications"

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OUTLINE

1. **INTRODUCTION**
2. THREE POINT ELICITATION
3. PRIOR DIRICHLET PROCESS CONSTRUCTION
4. BAYESIAN UPDATING USING FAILURE AND MAINTENANCE DATA
5. MAINTENANCE OPTIMIZATION
6. SELECTED REFERENCES

- **Maintenance optimization** has been a focus of research interest.

- **Dekker (1996) and Mazzuchi *et al.* (2014)** provide an elaborate review and analysis of applications of maintenance optimization models.

"Besides, many textbooks on operations research use replacement models as examples", Dekker (1996).

- **A main bottleneck** in the implementation of maintenance optimization procedures is **the determination of the life length distributions**.

- Due to **scarcity of good component failure data**, determination via known statistical estimation procedures is, in many cases, impossible. **Why is that?**

Answer:

- Scarcity of failure data is **inherent to an efficient preventive maintenance environment.** The complete component life cycle will rarely be observed.

- **Occurrence of many failures,** on the other hand, will lead to **equipment modification**, making past data obsolete.

Proposed Solution:

- One approach to overcome this scarcity of data is to determine the lifetime distribution based on the **use of expert judgment.**

- In the absence of data, **normative experts** are tasked with **specifying distributions that are consistent** with a **substantive expert's** judgment, whom **may not be statistically trained**.

- **To facilitate** such a situation, **integration of graphically interactive and statistical elicitation procedures** for distribution modeling has been a topic of research for **quite some time** with **some re-invigoration more recently**.

- See, DeBrota *et al.* (1989), Van Dorp (1989), AbouRizk *et al.* (1992), Van Noortwijk *et al.* (1992), Wagner and Wilson (1996).

- **More recently Van Dorp and Mazzuchi (2000), Garthwaite, Kadane and O'Hagan (2005) and Morris *et al.* (2014),** the latter developing a web-based distribution elicitation tool called 'MATCH', and **Shih N (2015).**

- Most of these indirect elicitation procedures **"fit" continuous distribution to the elicited expert judgement**, but do not match the expert judgement exactly, with the exception of **Van Dorp and Mazzuchi (2000) and Shih N (2015) who match two elicited quantiles uniquely to a beta distribution.**

- Herein, the elicitation of **lower and upper quantile estimates x_p and x_r** and **the most likely estimate η, $x_p < \eta < x_r$**, of a five-parameter **Generalized Two-Sided Power (GTSP) distribution** (Herrerías *et al.*, 2009) is proposed.

- The GTSP distribution with support (a, b) has prob. density function (pdf)

$$f(x|\Theta) = \mathcal{C}(\Theta) \times \begin{cases} \left(\frac{x-a}{\eta-a}\right)^{m-1}, & \text{for } a < x < \eta \\ \left(\frac{b-x}{b-\eta}\right)^{n-1}, & \text{for } \eta \leq x < b, \end{cases} \quad (1)$$

where $\Theta = \{a, \eta, b, m, n)$ and

$$\mathcal{C}(\Theta) = \frac{mn}{(\eta - a)n + (b - \eta)m}. \quad (2)$$

- The GTSP distribution was suggested as **a more flexible alternative to the classical beta distribution in the unimodal domain**.

- Moment Ratio (MR) diagrams **plots kurtosis β_2** against **"skewness"** $\sqrt{|\beta_1|}$ with convention that $\sqrt{|\beta_1|}$ retains **the sign of skewness β_1**.

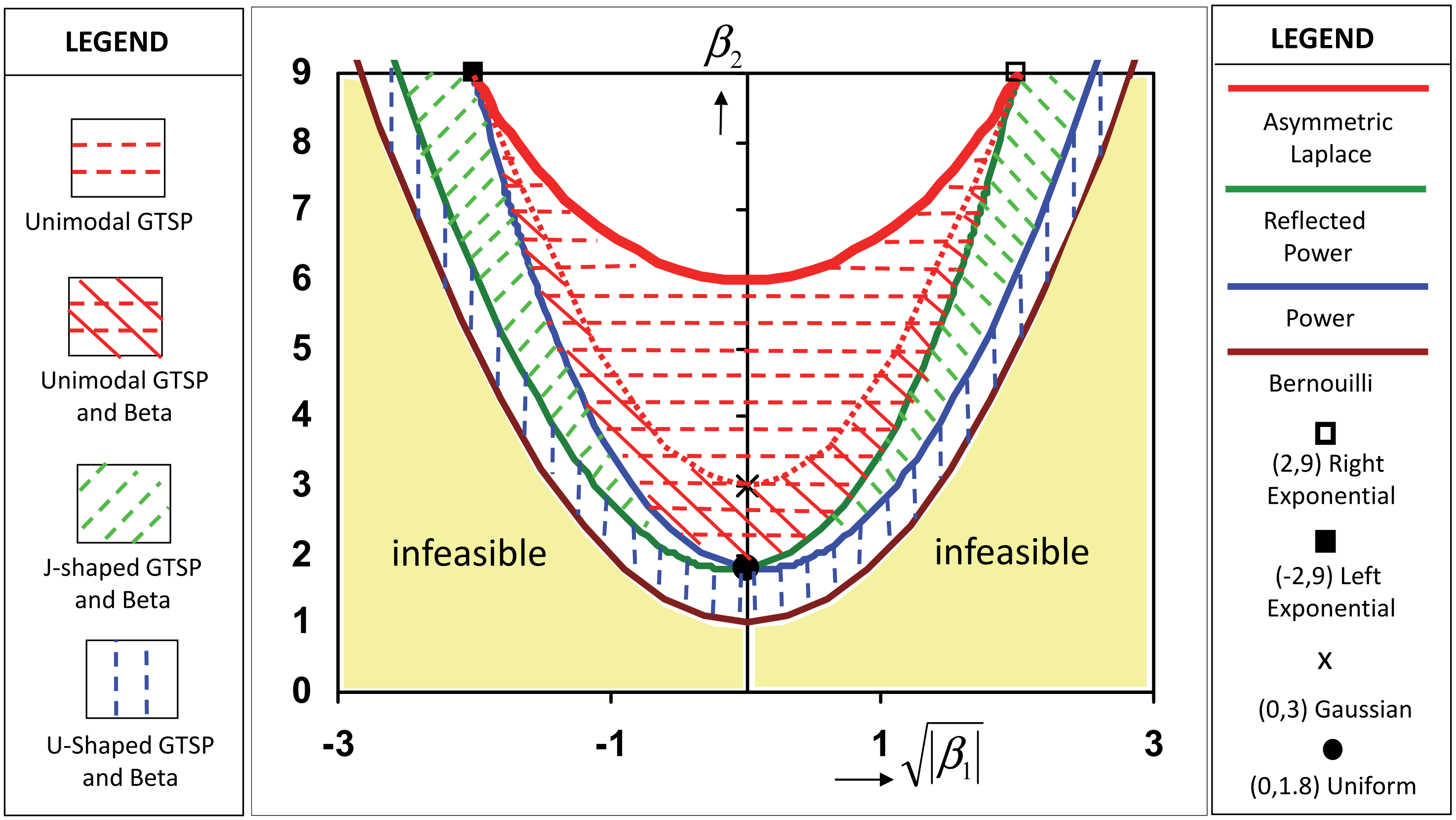

Figure 1. Moment Ratio ($\sqrt{\beta_1}$, β_2) coverage diagram for $GTSP$ (1) and beta pdfs.

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- **Given a fixed support (a, b)**, **chosen arbitrarily large**, standardize lower and upper quantile estimates x_p, x_r and most like value estimate η values **to values y_p, y_r and θ in $(0, 1)$** using **transformation $(x - a)/(b - a)$**.

- Utilizing that same linear transformation, the pdf (1) reduces to

$$f(y|m,n,\theta) = \frac{mn}{(1-\theta)m + \theta n} \times \begin{cases} \left(\frac{y}{\theta}\right)^{m-1}, & \text{for } 0 < y < \theta \\ \left(\frac{1-y}{1-\theta}\right)^{n-1}, & \text{for } \theta \leq y < 1. \end{cases} \quad (3)$$

$0 < \theta < 1, n, m > 0.$

- While the most likely value θ is elicited directly, the quantile estimates y_p, y_r are needed to **indirectly elicit the power-parameters m and n** of the pdf (3), hence the requirement $0 < y_p < \theta < y_r < 1$.

- From pdf (3) one directly obtains **the cumulative distribution function:**

$$F(y|\Theta) = \begin{cases} \pi(\theta, m, n)\left(\frac{y}{\theta}\right)^m, & \text{for } 0 \leq y < \theta \\ 1 - [1 - \pi(\theta, m, n)]\left(\frac{1-y}{1-\theta}\right)^n, & \text{for } \theta \leq y \leq 1, \end{cases} \quad (4)$$

with mode (or anti-mode) probability $Pr(X \leq \theta) = \pi(\theta, m, n) = \theta n/[(1-\theta)m + \theta n]$.

- **Given the quantile estimates y_p, y_r, the quantile constraints below** need to be solved to obtain **the power-parameters m and n** in (3), (4):

$$\begin{cases} F(y_p|\theta, m, n) = \pi(\theta, m, n)\left(\frac{y_p}{\theta}\right)^m = p, \\ F(y_r|\theta, m, n) = 1 - [1 - \pi(\theta, m, n)]\left(\frac{1-y_r}{1-\theta}\right)^n = r. \end{cases} \quad (5)$$

- It is proven that **the lower quantile constraint** in (5) defines **a unique implicit function** $m^{\bullet} = \xi(n)$, where $\xi(\cdot)$ is a **strictly increasing continuous concave function** in n, such that $\xi(n) \downarrow 0$ as $n \downarrow 0$ and $(m^{\bullet} = \xi(n), n)$ satisfies the first quantile constraint in (5) for all $n > 0$.

- As a result, when $n \downarrow 0$ the GTSP density $\boldsymbol{f(y|\xi(n), n, \theta)}$ converges to a Bernoulli distribution with **probability mass $\boldsymbol{p}$ at $\boldsymbol{y = 0}$** and **probability mass $\boldsymbol{1 - p}$ at $\boldsymbol{y = 1}$.**

- Finally, it is proven that the implicit function $\xi(n)$ has **the following tangent line at $\boldsymbol{n = 0}$** :

$$M(n|p,\theta) = n \times \frac{\theta}{1-\theta} \times \frac{1-p}{p}, \tag{6}$$

where **in addition** for all values of $n > 0$, $\boldsymbol{M(n|p,\theta) \geq \xi(n)}$.

$$y_p = 1/6, \theta = 4/15, y_r = 1/2,\ p = 0.2,\ r = 0.8$$
$$\Rightarrow m^* \approx 1.509 \text{ and } n^* \approx 2.840.$$

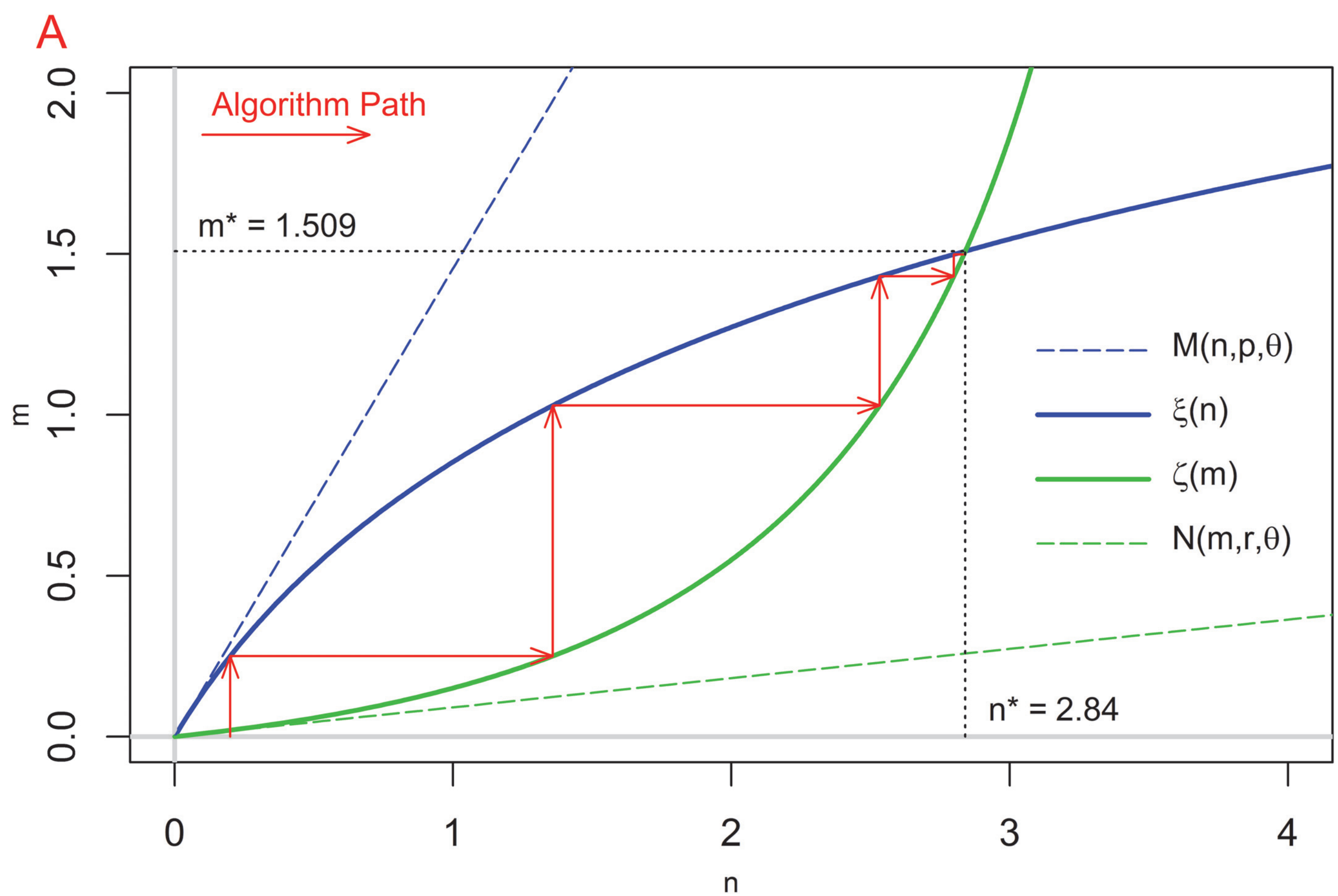


Figure 2. A: Implicit functions $\xi(n)$ and $\zeta(m)$ and algorithm path for the example data (10); B: $GTSP$ pdf solution (11); B: $GTSP$ cdf solution (11).

- It is proven that **the upper quantile constraint** in (5) defines **a unique implicit function** $n^{\bullet} = \zeta(m)$, where $\zeta(\cdot)$ is a **strictly increasing continuous concave function** in m, such that $\zeta(m) \downarrow 0$ as $m \downarrow 0$ and $(m, n^{\bullet} = \zeta(m))$ satisfies the second quantile constraint in (5) for all $m > 0$.

- As a result, when $m \downarrow 0$ the GTSP density $\boldsymbol{f(y|m, \zeta(m), \theta)}$ converges to a Bernoulli distribution with **probability mass $\boldsymbol{r}$ at $\boldsymbol{y = 0}$** and **probability mass $\boldsymbol{1 - r}$ at $\boldsymbol{y = 1}$**.

- Finally, it is proven that the implicit function $\zeta(m)$ has **the following tangent line at $\boldsymbol{m = 0}$** :

$$N(m|r, \theta) = m \times \frac{1-\theta}{\theta} \times \frac{r}{1-r}, \tag{7}$$

where for all values of $\boldsymbol{m > 0, \; N(m|r, \theta) \geq \zeta(m).}$

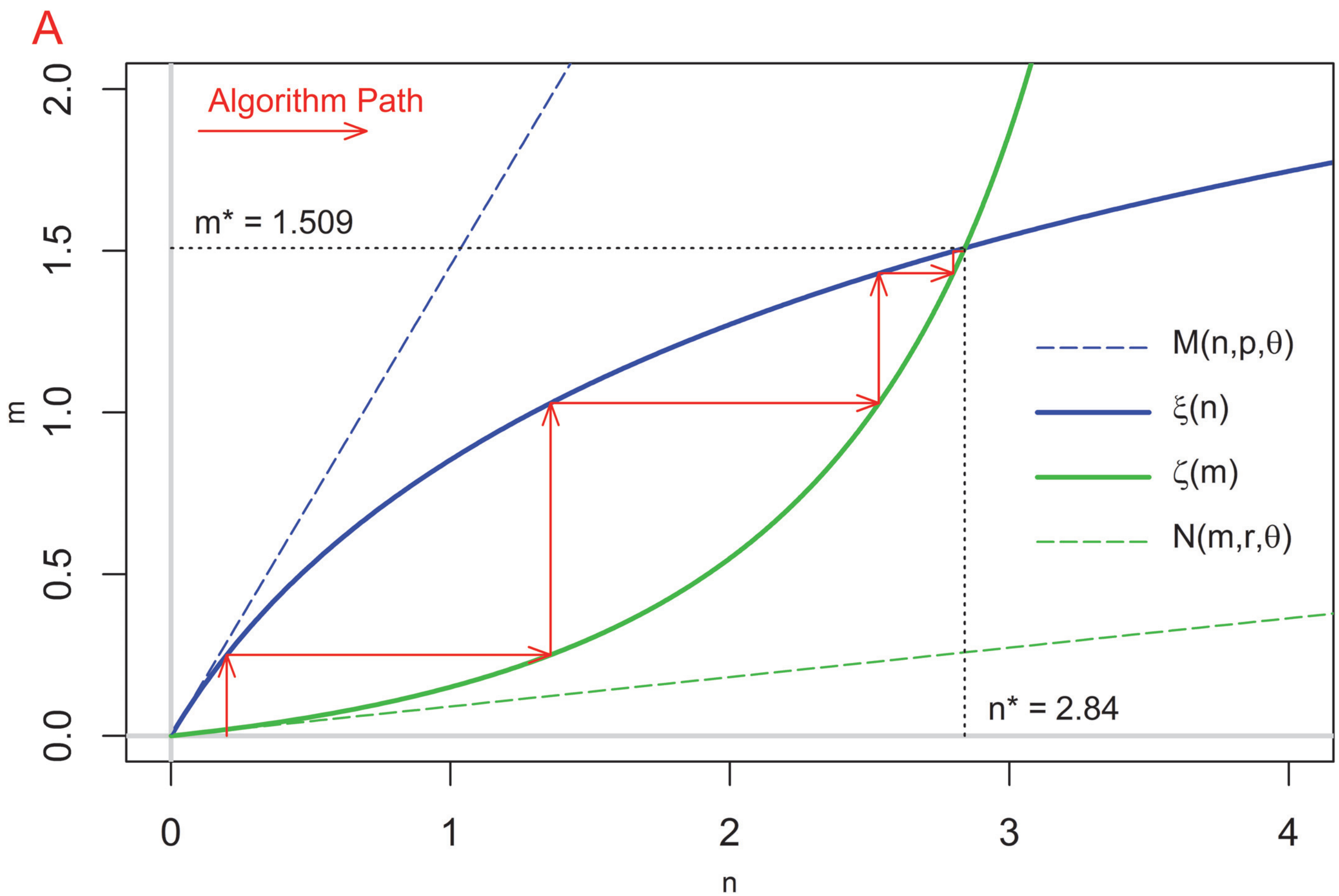
- From these conditions it follows that **the quantile constraint set (7)** has **a unique solution (m^*, n^*) where $m^*, n^* > 0$.**

- **The unique solution $m^{\bullet} = \xi(n)$ for a fixed value $n > 0$** may be solved using, e.g., GoalSeek in Microsoft Excel. **The unique solution $n^{\bullet} = \zeta(m)$ may be solved for a fixed value of $m > 0$** in a similar manner.

- **The following algorithm** now solves for **(m^*, n^*) where $m^*, n^* > 0$.**

  Step 1: Set $n^{\bullet} = \delta > 0$ (arbitrarily small).
  Step 2: Calculate $m^{\bullet} = \xi(n^{\bullet})$ (satisfying first quantile constraint in (5)).
  Step 3: Calculate $n^{\bullet} = \zeta(m^{\bullet})$ (satisfying second quantile constraint in (5)).
  Step 4: If $\left|\pi(\theta, m^{\bullet}, n^{\bullet})\left(\frac{y_p}{\theta}\right)^{m^{\bullet}} - p\right| < \epsilon$ **Then** Stop **Else Goto** Step 2.

$$y_p = 1/6, \theta = 4/15, y_r = 1/2,\ p = 0.2,\ r = 0.8$$
$$\Rightarrow m^* \approx 1.509 \text{ and } n^* \approx 2.840.$$



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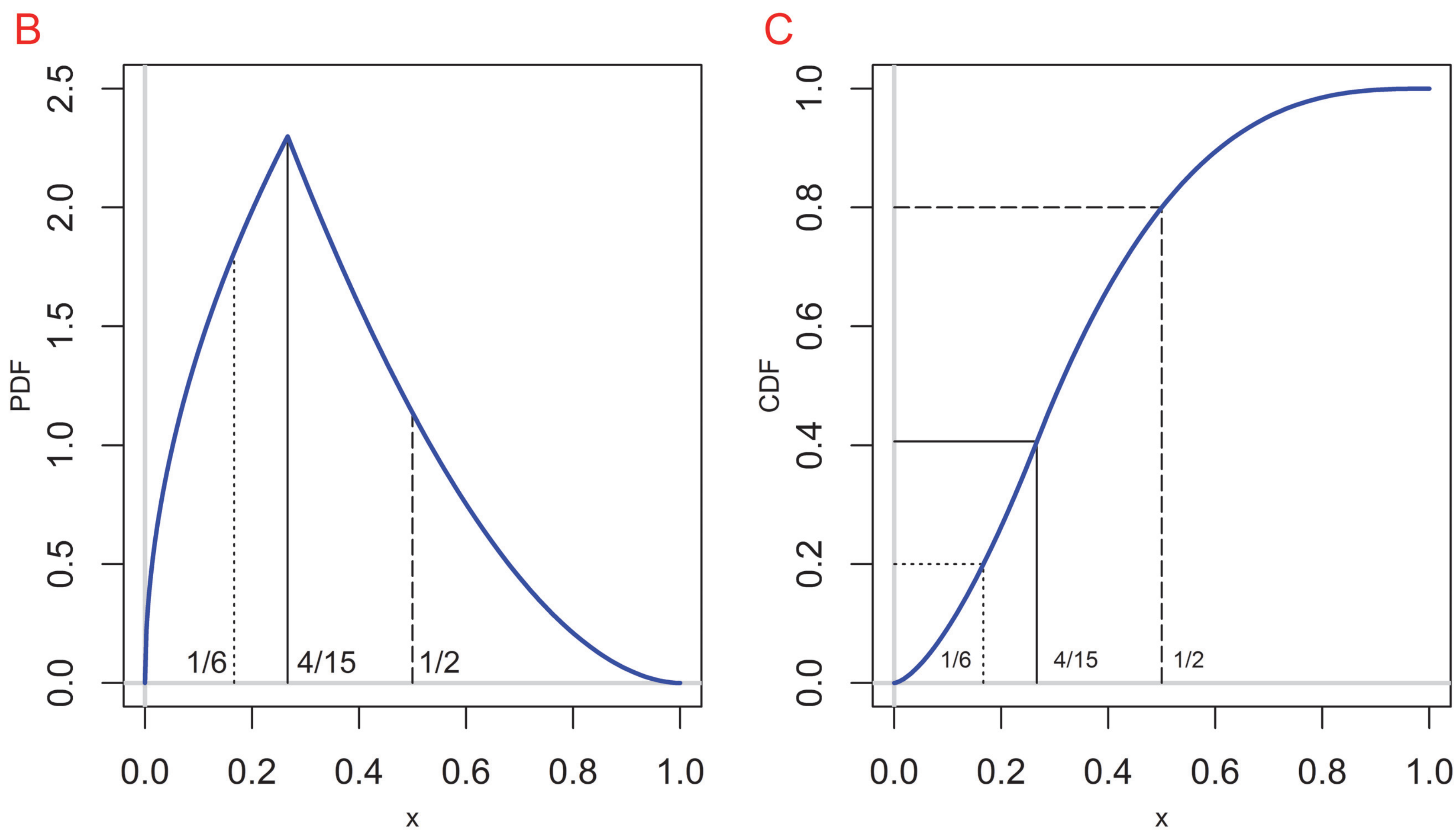

Figure 2. B: $GTSP$ **pdf solution (11); B:** $GTSP$ **cdf solution (11).**

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- **Aim:** Use elicited expert life time distributions $F_e(x)$, $e = 1, \ldots, E$ to **specify the prior parameters of a Dirichlet Process**. **A Dirichlet process (Ferguson, 1973)** may be used to define a distribution for the cdf $F(x)$ for every time $x \in (0, \infty) = \mathbb{R}^+$. **Below a 5 step procedure is demonstrated.**

- **Ferguson (1973)** showed that for a DP with parameter measure $\alpha(\mathcal{A}) > 0$, $\mathcal{A} \subset \mathbb{R}^+$, $\boldsymbol{F(x) \sim Beta(\,\alpha\{(0,x)\},\,\alpha\{[x,\infty)\})}$. Thus with

$$\boldsymbol{\alpha(\mathbb{R}^+) = \alpha\{(0,x)\} + \alpha\{[x,\infty)\}}$$

we have

$$E[F(x)|\alpha(\,\cdot\,)] = \frac{\boldsymbol{\alpha\{(0,x)\}}}{\boldsymbol{\alpha(\mathbb{R}^+)}},$$

$$V[F(x)|\alpha(\,\cdot\,)] = \frac{\boldsymbol{\alpha\{(0,x)\}} \times \{\boldsymbol{\alpha(\mathbb{R}^+)} - \boldsymbol{\alpha\{(0,x)\}}\}}{\{\boldsymbol{\alpha(\mathbb{R}^+)}\}^2\{\boldsymbol{\alpha(\mathbb{R}^+)} + 1\}}.$$

- **Step 1:** Set $F_d(x) = \frac{1}{E}\sum_{e=1}^{E} F_e(x) = \overline{F(x)}$ using **an equal-weighted linear opinion (see, e.g. Cooke, 1991)** since in Bayesian context data, hopefully, eventually outweighs the prior expert information.

Table 1. Illustrative example A: Support $[0, 30]$ B: Support $[0, 100]$

A	EXPERT 1	EXPERT 2	EXPERT 3		EXPERT 1	EXPERT 2	EXPERT 3
a	0	0	0	a	0	0	0
p	0.2	0.2	0.2	p	0.2	0.2	0.2
r	0.8	0.8	0.8	r	0.8	0.8	0.8
b	30	30	30	b	1	1	1
x_p	5	2	6	y_p	1/6	1/15	1/5
η	8	4	9	θ	4/15	2/15	3/10
x_r	15	7	12	y_r	1/2	7/30	2/5
m	1.504	1.269	2.328	m	1.504	1.269	2.328
n	2.838	7.733	5.755	n	2.838	7.733	5.755

B	EXPERT 1	EXPERT 2	EXPERT 3		EXPERT 1	EXPERT 2	EXPERT 3
a	0	0	0	a	0	0	0
p	0.2	0.2	0.2	p	0.2	0.2	0.2
r	0.8	0.8	0.8	r	0.8	0.8	0.8
b	100	100	100	b	1	1	1
x_p	5	2	6	y_p	0.05	0.02	0.06
η	8	4	9	θ	0.08	0.04	0.09
x_r	15	7	12	y_r	0.15	0.07	0.12
m	1.592	1.289	2.363	m	1.592	1.289	2.363
n	13.397	29.565	26.029	n	13.397	29.565	26.029

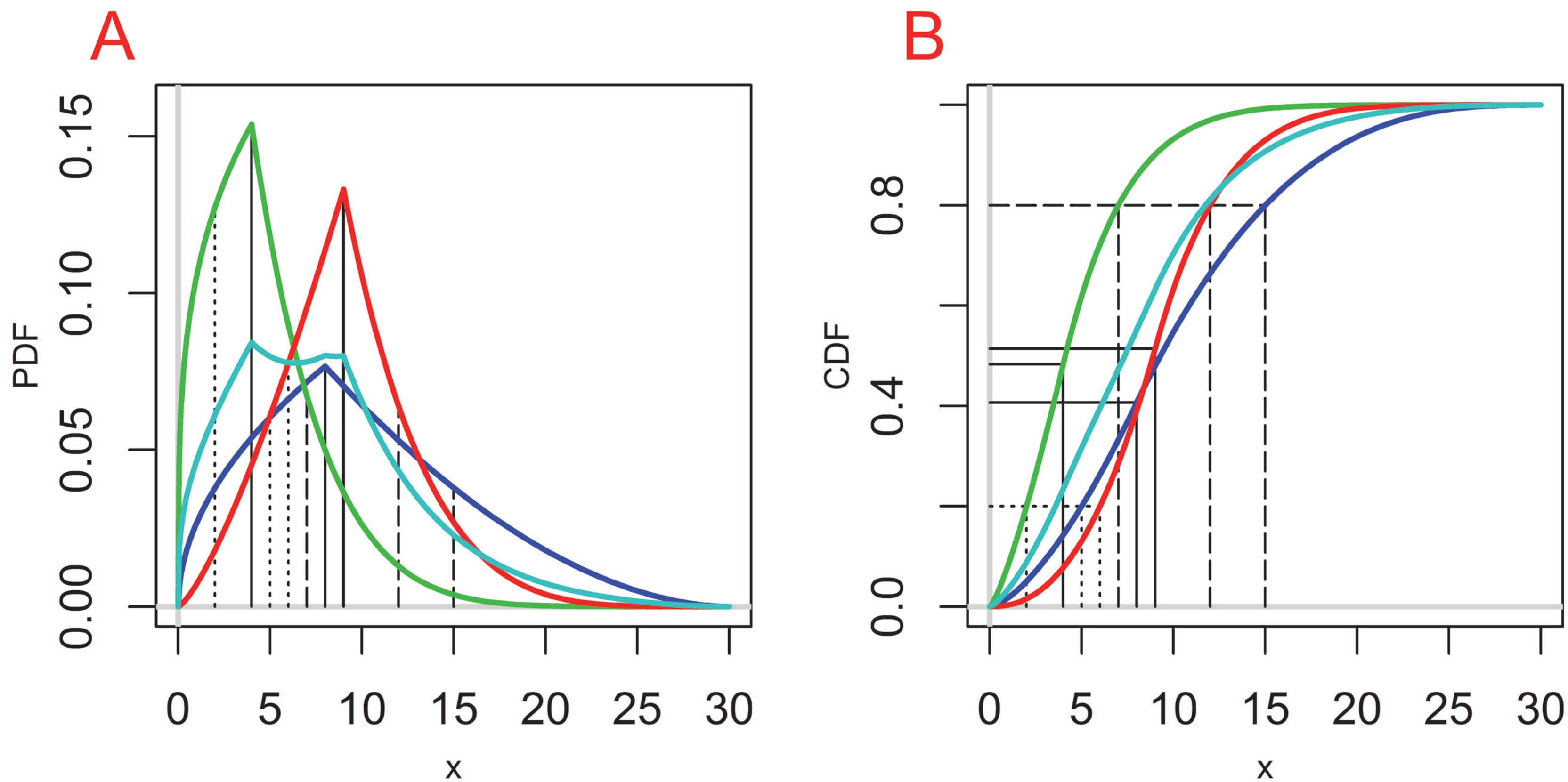

Figure 3. GTSP distribution for the expert data in Table 1. Expert 1's distribution in dark blue, Expert 2's distribution in green, Expert 3's distribution in red, equi-weight mixture distribution in light blue.

- **Step 2:** Fit **Generalized Trapezoidal** cdf $F(t|\Theta)$ to $F_d(t)$ (although not required for prior DP construction, but **provides parametric convenience**).

- **The Generalized Trapezoidal cdf with support (a,b)** is given by:

$$F(x|\Theta)=\begin{cases}\frac{2\alpha(b-a)n_3}{2\alpha(\eta_1-a)n+(\alpha+1)(\eta_2-\eta_1)mn+2(b-\eta_2)m}\left(\frac{x-a}{\eta_1-a}\right)^m, & \text{for } a\leq x<\eta_1\\ \frac{2\alpha(b-a)n_3+2(x-b)n_1n_3\left\{1+\frac{(\alpha-1)}{2}\frac{(2c-b-x)}{(c-b)}\right\}}{2\alpha(\eta_1-a)n+(\alpha+1)(\eta_2-\eta_1)mn+2(b-\eta_2)m}, & \text{for } \eta_1\leq x<\eta_2\\ 1-\frac{2(d-c)n_1}{2\alpha(\eta_1-a)n+(\alpha+1)(\eta_2-\eta_1)mn+2(b-\eta_2)m}\left(\frac{d-x}{d-\eta_2}\right)^n, & \text{for } \eta_2\leq x<b.\end{cases}$$

- Set $(a,b)=(0,30)$, set $\eta_1=4$ (**the smallest elicited most likely estimate in Table 1**) and set $\eta_2=9$ (**the largest most likely estimate in Table 1**).

- Solve for **GT parameters α, m and n** of the using **a least squares procedure between** the equi-weight mixture cdf and the GT cdf, resulting in

$$\alpha=1.056, m=1.390, n=4.464.$$

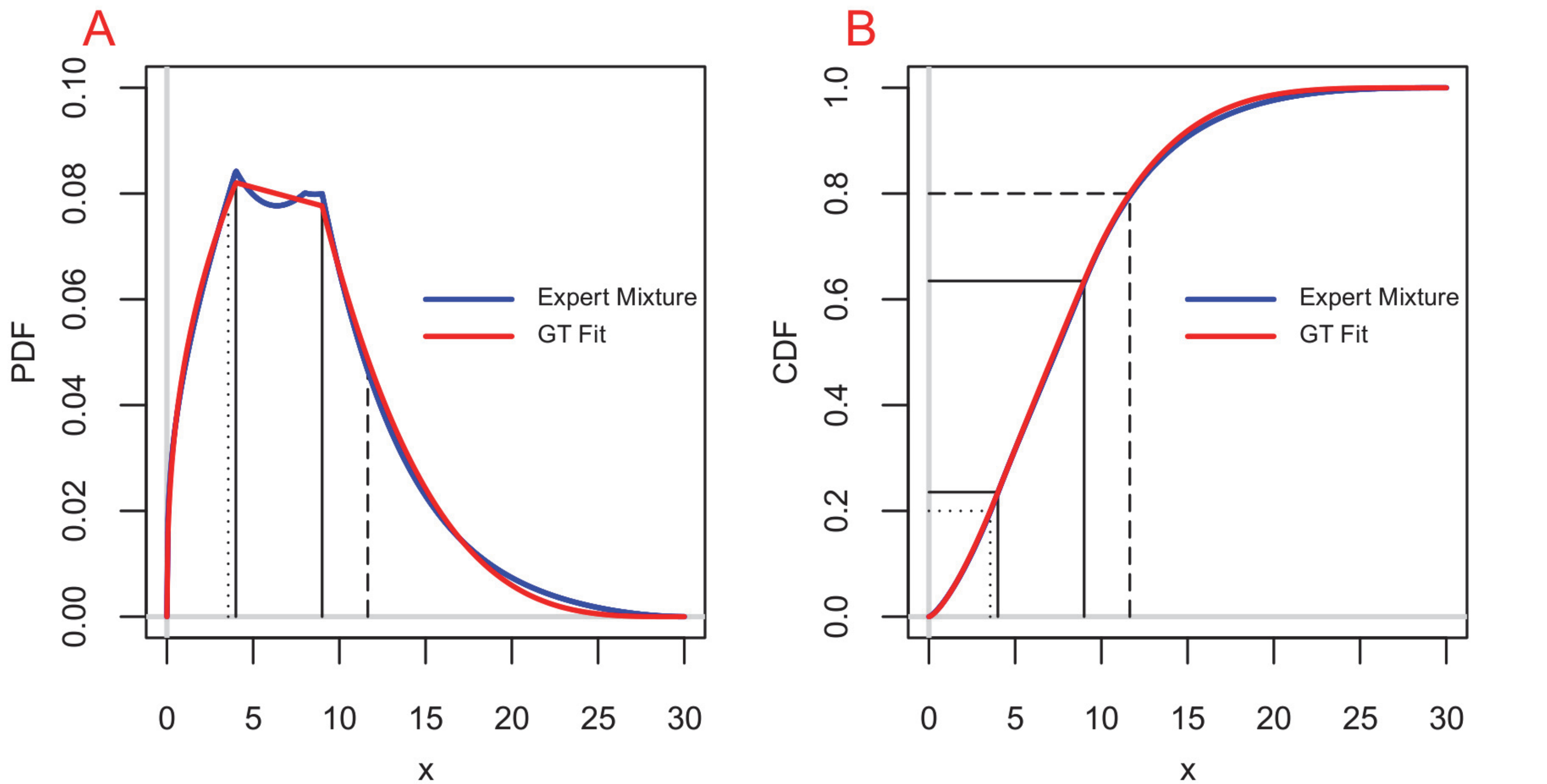

Figure 4. Equi-weight mixture distribution (in blue), GT fit to the mixture distribution (in light green). A: pdfs, B: cdfs.

- **Step 3:** Encapsulate prior knowledge in the Dirichlet Process (DP) by setting:

$$\boldsymbol{\alpha}[(0, \boldsymbol{x})] = \boldsymbol{\alpha}(\mathbb{R}^{+}) \times F(x|\Theta).$$

- This yields for the **Dirichlet Process**:

$$E[F(x)|\alpha(\,\cdot\,)] = F(x|\Theta), \text{i.e } \textbf{the fitted GT cdf} \tag{8}$$
$$V[F(x)|\alpha(\,\cdot\,)] = \frac{F(x|\Theta)\times[1-F(x|\Theta)]}{\boldsymbol{\alpha}(\mathbb{R}^{+})+1}.$$

- Observe that $\boldsymbol{\alpha}(\mathbb{R}^{+})$ is positive constant that **drives the variance in $F(x)$**.

- **Step 4:** Evaluate x^* that maximizes

$$\widehat{V}[F(x)] = \frac{1}{E-1}\sum_{e=1}^{E}[F_e(x|a,\eta,b,m,n) - F(x|\Theta)]^2, \tag{9}$$

- **Step 5:** Solve $\boldsymbol{\alpha}(\mathbb{R}^{+})$ from (9) by setting

$$V[F(x^*)|\alpha(\,\cdot\,)] = \widehat{V}[F(x^*)], \tag{10}$$

$$x^* = 6.462 \text{ with } \widehat{V}[F(x^*)] = 0.0824 \text{ and } E[F(x^*|\Theta)] = 0.435$$
$$\Rightarrow \boldsymbol{\alpha(\mathbb{R}^+) \approx 1.9832.}$$

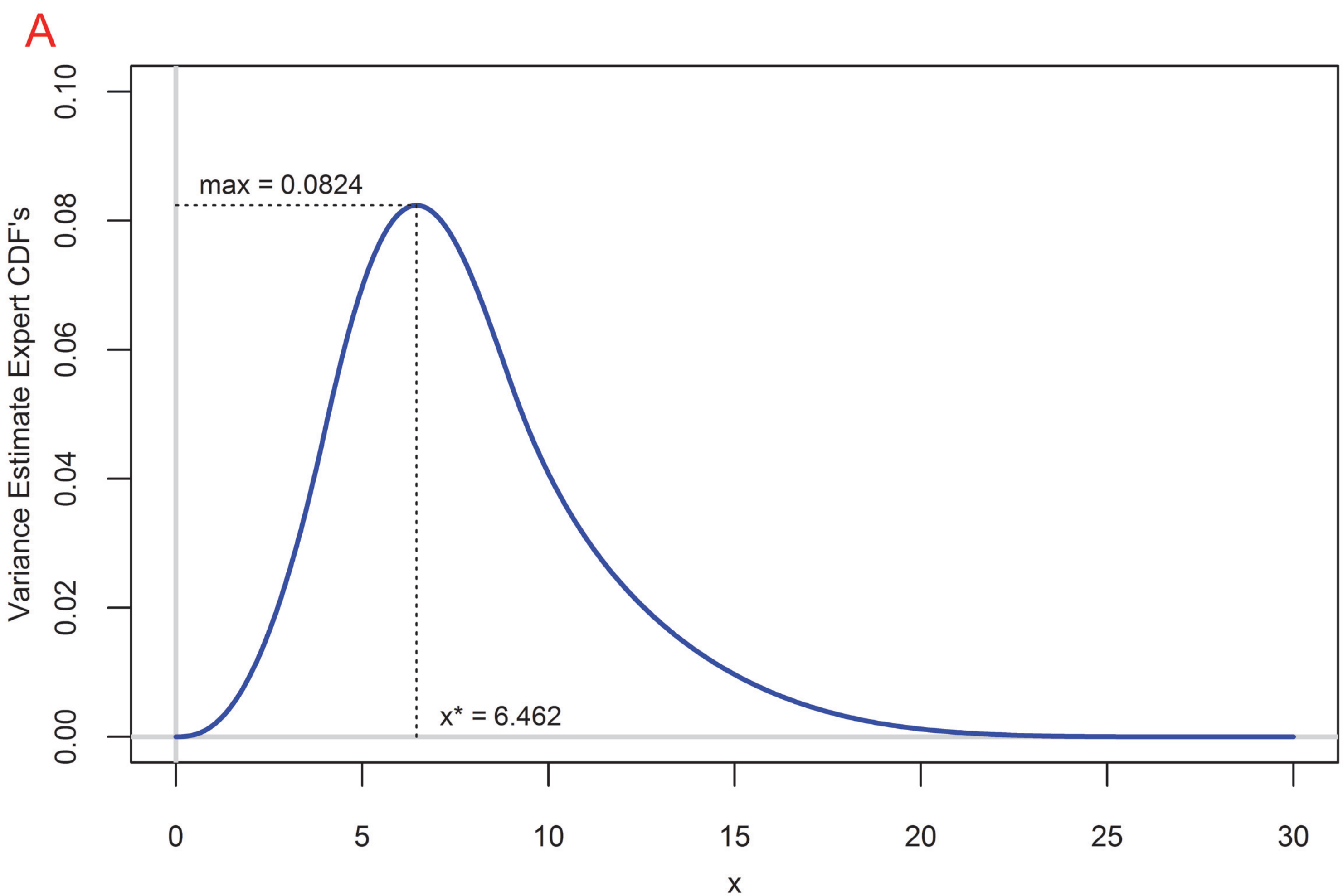

Figure 5. A: Plot of $\widehat{V}[F(x)]$ given by (19) for the example data in Table 1

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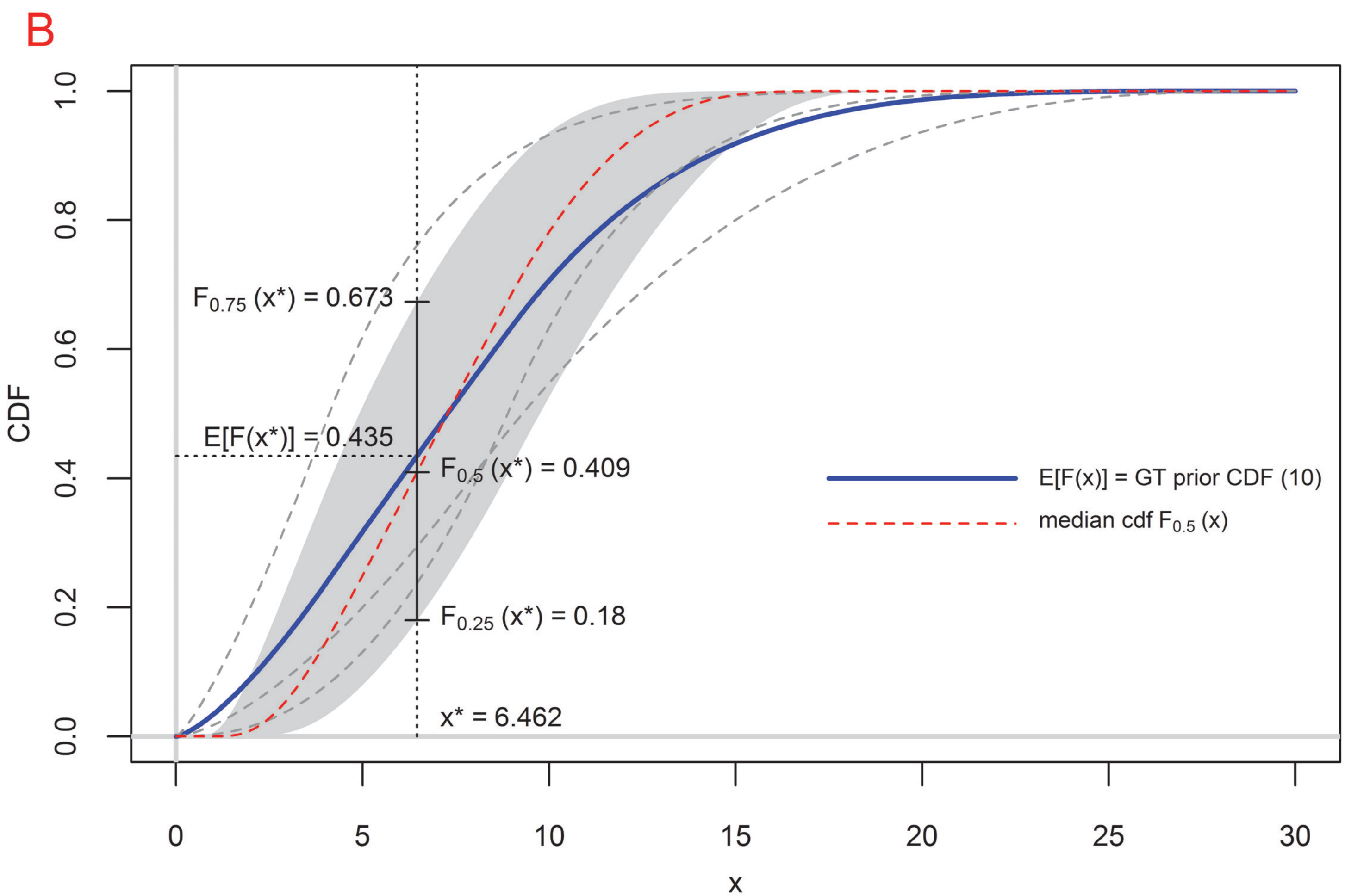

Figure 5. B: Summary of the resulting uncertainty in the prior Dirichlet Process for the cdf $F(x)$.

OUTLINE

- **Failure Data:** $(\boldsymbol{n_x}, \underline{\boldsymbol{x}}) \equiv (x_{(1)}, \ldots x_{(n_x)})$ a sample of ordered fail. times x_j.

- **Ferguson (1973)'s** main theorem entails that posterior distribution **given observed failure data** $(\boldsymbol{n_x}, \underline{\boldsymbol{x}})$ for $x_{(i)} \leq x < x_{(i+1)}, i = 1, \ldots, n_x$ is:

$$[F(x)|(n_x, \underline{x})] \sim$$
$$Beta(\boldsymbol{\alpha}(\mathbb{R}^+) \times \boldsymbol{F}(\boldsymbol{x}|\boldsymbol{\Theta}) + \boldsymbol{i}, \boldsymbol{\alpha}(\mathbb{R}^+) \times [1 - \boldsymbol{F}(\boldsymbol{x}|\boldsymbol{\Theta})] + \boldsymbol{n_x} - \boldsymbol{i})$$

with **posterior expectation**

$$E[F(x)|\alpha(\cdot), (n_x, \underline{x})] = \boldsymbol{\lambda_{n_x}} \boldsymbol{F}(\boldsymbol{x}|\boldsymbol{\Theta}) + (1 - \boldsymbol{\lambda_{n_x}})\widehat{\boldsymbol{F}}_{\boldsymbol{n_x}}[\boldsymbol{x}|(\boldsymbol{n_x}, \underline{\boldsymbol{x}})],$$

where

$$\lambda_{n_x} = \frac{\alpha(\mathbb{R}^+)}{\alpha(\mathbb{R}^+) + n_x},$$
$$\widehat{F}_{n_x}[x|(n_x, \underline{x})] = \frac{i}{n_x} \text{ for } x_{(i)} \leq x < x_{(i+1)}, i = 1, \ldots, n_x,$$

and $x_{(0)} \equiv 0$, $x_{(n_x+1)} \equiv \infty$. **Thus $\boldsymbol{\alpha}(\mathbb{R}^+)$ is a "virtual sample size".**

$$\alpha(\mathbb{R}^+) \approx 1.9832, n_x = 5, \Rightarrow \lambda_{n_x} \approx 0.2840 \Rightarrow \textbf{prior obtains 28.40\% weight}$$

$$x_{(1)} = 4, x_{(2)} = 10, x_{(3)} = 11, x_{(4)} = 13, x_{(5)} = 15$$

A

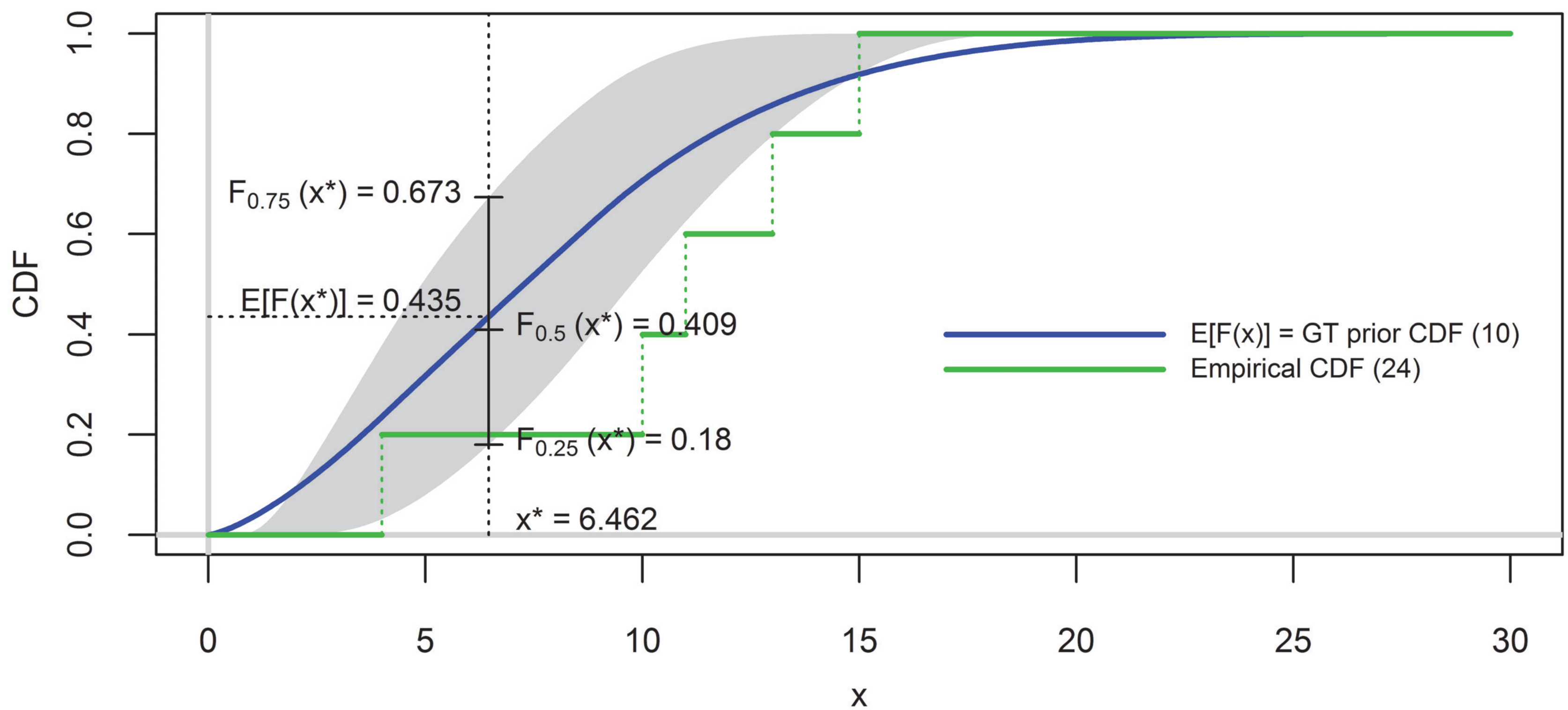

Figure 6. Comparison of the empirical cdf $\widehat{F}_{n_x}[x|(n_x, \underline{x})]$ with the prior and posterior estimates from the DP's for the cdf $F(\cdot)$ and their inter quartile ranges. A: Prior cdf $F(x|\Theta)$, B: posterior cdf $E[F(x)|\alpha(\cdot), (n_x, \underline{x})]$ given failure data.

$$\alpha(\mathbb{R}^+) \approx 1.9832, n_x = 5, \Rightarrow \lambda_{n_x} \approx 0.2840 \Rightarrow$$ **prior obtains 28.40% weight**

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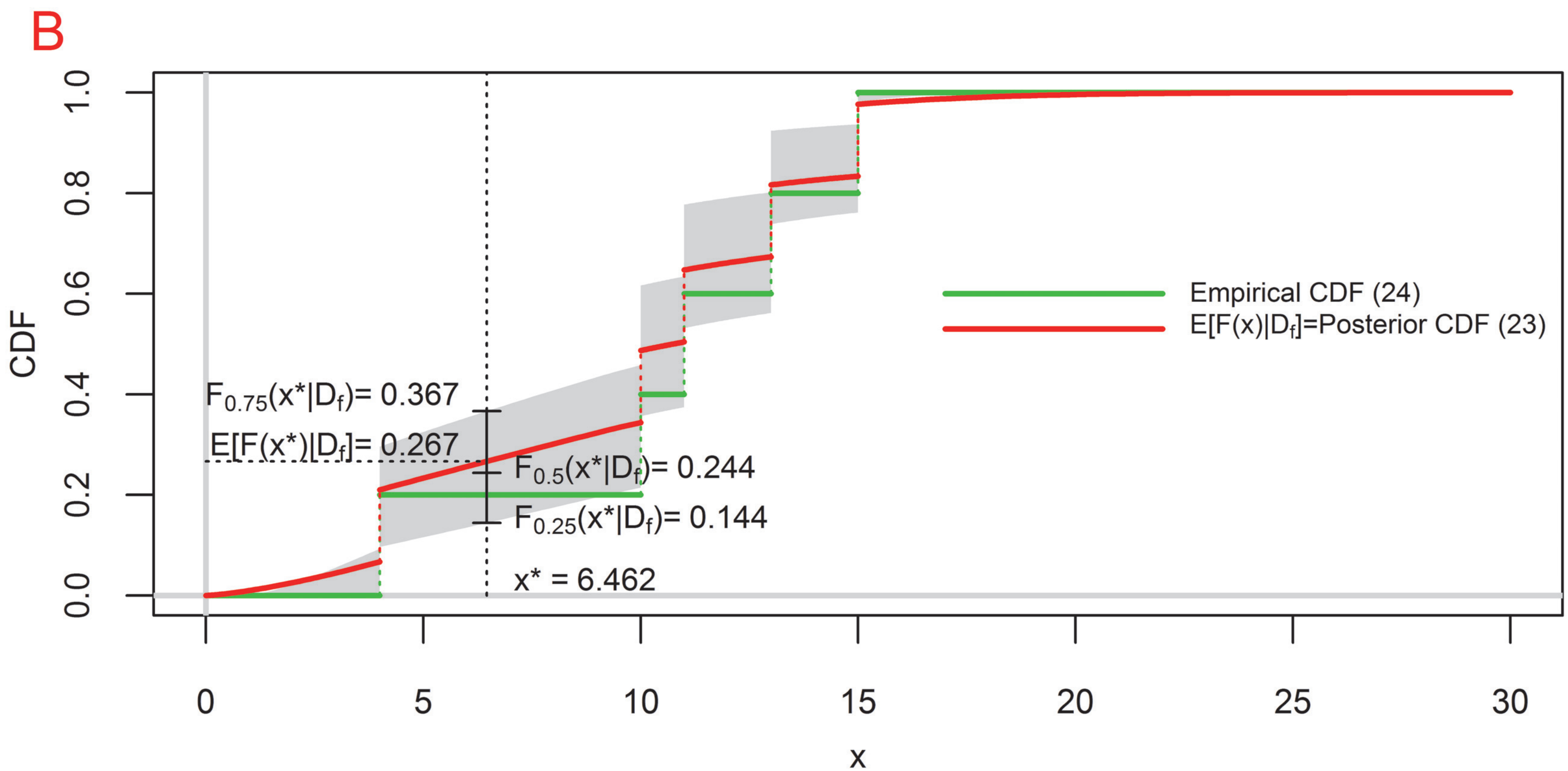

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- **Maintenance Data:**

$$[n_c, (\underline{\gamma}, \underline{c})] \equiv [(\gamma_1, c_{(1)}), \ldots (\gamma_{n_c}, c_{(n_c)})]$$

where $(\gamma_j, c_{(j)})$ indicates that t**he component was removed from service γ_j times** at **censor time $c_{(j)}$** to **be preventively maintained**.

- Join **the failure data $(n_x, \underline{x})$** with **maintenance data $[n_c, (\underline{\gamma}, \underline{c})]$:**

$$[m_z, n_z, (\underline{\delta}, \underline{z})] = [(\delta_1, z_{(1)}), \ldots, (\delta_{m_z}, z_{(m_z)})] \ ,$$

$$\boldsymbol{m_z} = n_x + n_c, \ \boldsymbol{n_z} = \boldsymbol{n_x} + \sum_{j=1}^{n_c} \gamma_i,$$

$$\delta_j = \begin{cases} 1, & z_{(j)} \in (x_{(1)}, \ldots x_{(n_x)}), \\ \gamma_j, & (\gamma_j, z_{(j)}) \in [(\gamma_1, c_{(1)}), \ldots, (\gamma_2, c_{(n_c)})], \end{cases}$$

- **Susarla and Van Ryzin (1976)** derived the following expression for the **posterior moments of the survival function $S(x) = 1 - F(x)$** for $c_{(k)} \leq x < c_{(k+1)}$, $k = 0, \ldots, n_c$, $c_{(0)} \equiv 0, c_{(n_c+1)} \equiv \infty$:

$$E[S^p(x)|\Psi] = \prod_{s=0}^{p-1} \left[\frac{\alpha\{(x,\infty)\} + s + n^+(x)}{\alpha(\mathbb{R}^+) + s + n_z} \right] \times \xi\{x, s|\alpha(\cdot), [n_c, (\underline{\gamma}, \underline{c})]\}, p = 1, 2, \ldots$$

where $\Psi = \{\alpha(\cdot), [m_z, n_z, (\underline{\delta}, \underline{z})],$

$$\xi\{x, s|\alpha(\cdot), [n_c, (\underline{\gamma}, \underline{c})]\} = \prod_{j=1}^{k} \frac{\alpha(\mathbb{R}^+) \times S(c_{(j)}|\Theta) + s + n(c_{(j)})}{\alpha(\mathbb{R}^+) \times S(c_{(j)}|\Theta) + s + n(c_{(j)}) - \gamma_j}$$

$[m_z, n_z, (\underline{\delta}, \underline{z})]$ is the joint failure and maintenance data, **$\alpha(\cdot)$ is the parameter measure of a Dirichlet process**, and finally

$$n^+(x) = \sum_{\{i:z_{(i)}>x\}} \delta_i, \text{ and } n(x) = \sum_{\{i:z_{(i)}\geq x\}} \delta_i.$$

- **Substitution of $p = 1$** and $\alpha(\cdot)$ yields for $c_{(k)} \leq x < c_{(k+1)}, k = 0, \ldots, n_c$, $c_{(0)} \equiv 0,\ c_{(n_c+1)} \equiv \infty$ an **alternative expression for $E[S(x)|\Psi]$** utilizing

$$\widehat{S}_{n_z}\{x|[m_z, n_z, (\underline{\delta}, \underline{z})]\} = \frac{n^+(x)}{n_z}$$

which can be intepreted **as *lower-bound* definition for the empirical survival function** (since a maintenance removal is "counted" above as a failure point):

$$E[S(x)|\Psi] = \xi\{x, 0|\alpha(\cdot), [n_c, (\underline{\gamma}, \underline{c})]\} \times \left\{\lambda_{n_z} S(x|\Theta) + (1 - \lambda_{n_z})\widehat{S}_{n_z}\{x|[m_z, n_z, (\underline{\delta}, \underline{z})]\}\right\}.$$

where

$$\lambda_{n_z} = \frac{\alpha(\mathbb{R}^+)}{\alpha(\mathbb{R}^+) + n_z}, S(x|\Theta) = 1 - F(x|\Theta),$$

and $F(x|\Theta)$ is the prior GT cdf (13).

Joining failure $x_{(1)} = 4, x_{(2)} = 10, x_{(3)} = 11,\ x_{(4)} = 13,\ x_{(5)} = 15$ with $[n_c, (\underset{\sim}{\gamma}, \underline{c})] \equiv \{(4,3),(3,6),(2,9),(1,12)\} \Rightarrow n_c = 4, n_z = 15, \lambda_{n_z} \approx 0.1167.$

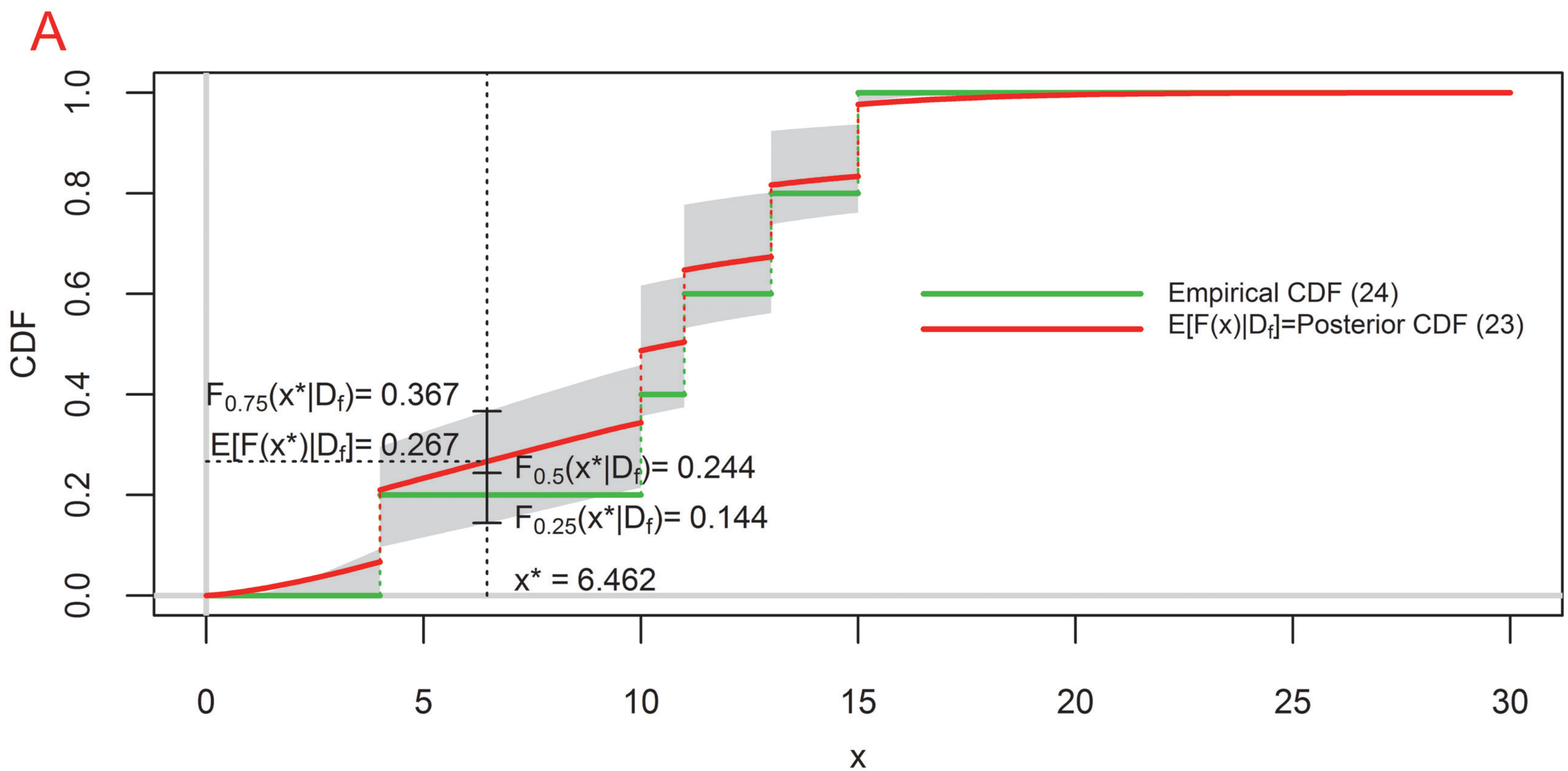

Figure 7. Comparison of the empirical cdf $\widehat{F}_{n_x}[x|(n_x,\underline{x})]$ with posterior estimates and their inter quartile ranges. A: posterior cdf $E[F(x)|\alpha(\cdot),(n_x,\underline{x})]$, B: posterior cdf $E[F(x)|\Psi] = 1 - E[S(x)|\Psi]$ given failure data and maintenance data.

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$$\text{Joining failure } x_{(1)}=4, x_{(2)}=10, x_{(3)}=11,\ x_{(4)}=13,\ x_{(5)}=15 \text{ with}$$
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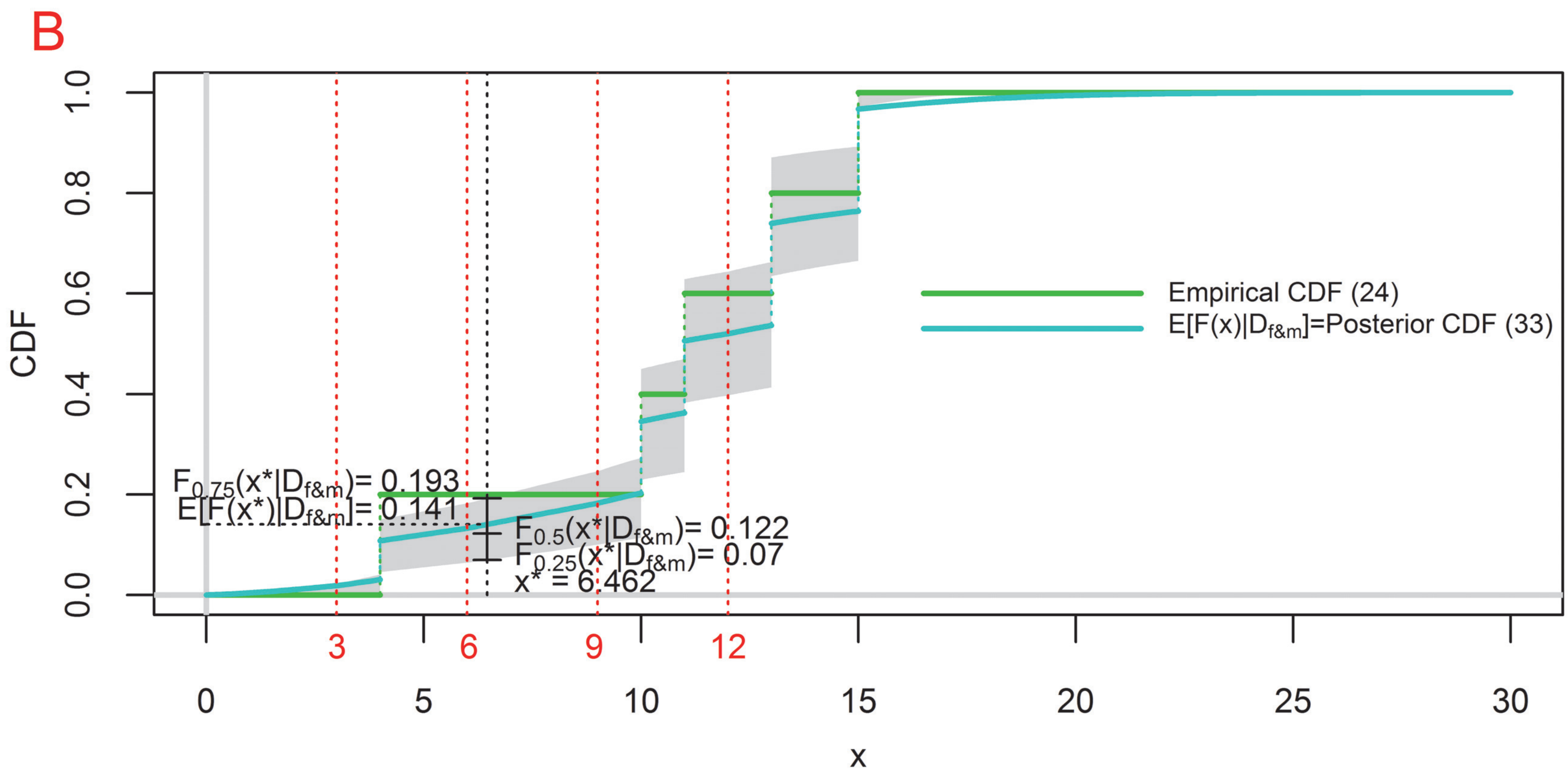

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- The latter IQRs were obtained **by fitting beta distributions to the first two posterior moments**, while verifying that **the third and fourth moments of these beta fitted distributions equal the third and fourth posterior moments**, **upto three decimal places.**

- **Susarla and Van Ryzin (1976)** assumed random observations $\boldsymbol{Z_i} = \textbf{min}(\boldsymbol{X_i}, \boldsymbol{C_i})$, where the X_i random failure times are $i.i.d$, and the C_i's are random censoring times also independent from the X_i's.

- **The $\boldsymbol{C_i}$ random variables** are assumed to **be mutually independent**, but do not have to be identically distributed and **could be degenerate implying fixed maintenance times**.

- **In case of no censoring $\boldsymbol{n_z = n_x}$**, $\widehat{S}_{n_z}[x|\{n_z, (\underline{\delta}, \underline{z})\}]$ reduces to **the empirical survival function** given failure data $\{n_x, \underline{x}\}$, and **the product term reduces to the value 1** since $k = 0$ in the no censoring case. Hence, the **Susarla and Van Ryzin (1976) formula reduces to Ferguson (1973)'s**.

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- **A basic model** within the context of **maintenance optimization** is **the block replacement model** where **at age x** of the component a prescheduled preventive maintenance action is carried out.

- One obtains for **the long term average cost per unit time:**

$$[g(x)|\Lambda(x)] = \frac{\mathcal{K}_p + \mathcal{K}_f \times \Lambda(x)}{x},$$

where $\Lambda(x) \equiv$ the expected number of failures during the maintenance cycle x, $\mathcal{K}_f$ **are the expected failure cost** and $\mathcal{K}_p$ **are the preventive maintenance cost**. As $\mathcal{K}_f$ is unplanned it is assumed that $\mathcal{K}_f > \mathcal{K}_p$.

- **Under a minimal repair assumption** the failure process

$$N(x) \equiv \# \text{ Failures in the interval from } [0, x],$$

can be described **as non-homogenous Poisson process** with with **intensity/failure rate function** $\lambda(u),\ u > 0$.

- Given **intensity function** $\lambda(u),\ u>0$ one obtains **mean value function**:

$$\Lambda(x)=\int_0^x \lambda(u)du \text{ and } Pr(N(x)=k)=\frac{\{\Lambda(x)\}^k}{k!}exp\{-\Lambda(x)\}.$$

- **Denoting $X_1 \sim F(\cdot)$** to be **the failure time of *the first failure*** with **time-to-failure cdf $F(\cdot)$**, one obtains the following equivalency:

$$Pr(N(x)=0)=exp(-\Lambda(x))\equiv Pr(X_1>x)=1-F(x).$$

- Hence $\boldsymbol{\Lambda(x)=-\ln\{1-F(x)\}}$ and

$$[g(x)|\Lambda(x)]\equiv[g(x)|F(x)]=\frac{K_p-K_f\times \ln\{1-F(x)\}}{x}.$$

- **The optimal maintenance interval** (**given the cdf $F(\cdot)$**) is that time point $x^\bullet$ for which $[g(x^\bullet)|F(x^\bullet)]$ is minimal.

- Given that in a ***Dirichlet Process*** the **cdf $F(x)$ a is random variable for every fixed value of x** $\Leftrightarrow$ the LTAC per unit time **$[g(x)|F(x)]$ is too a random variable!**

$\mathcal{K}_f = 20$ and $\mathcal{K}_p = 2$, i.e. a failure is **ten times more costly** than a preventative maintenance action.

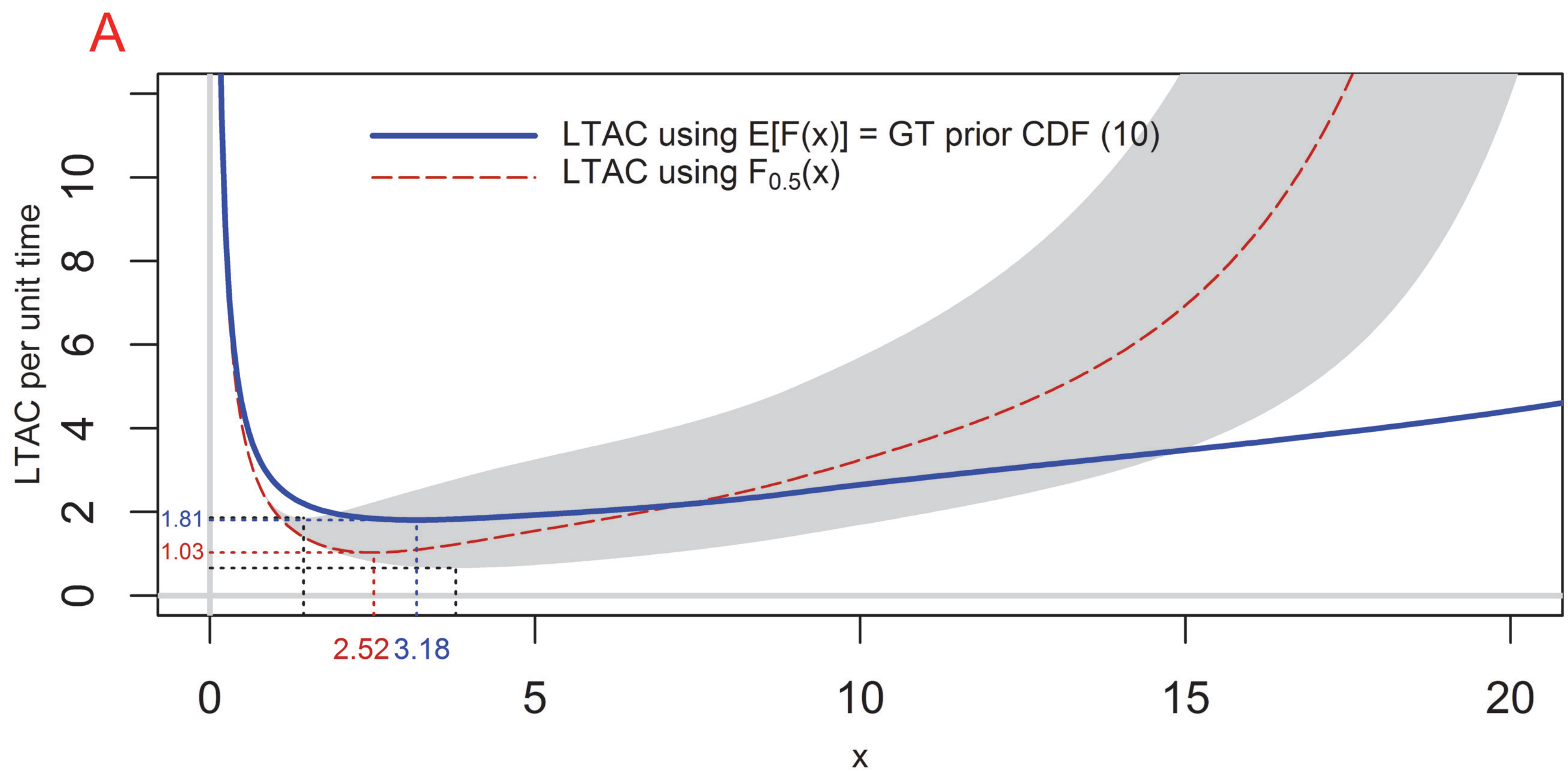

Figure 8. Comparison of prior and posterior estimates for the LTAC per unit time $[g(x) \mid F(x)]$ with $\mathcal{K}_f = 20$ and $\mathcal{K}_p = 2$ and their inter quartile ranges. A: prior estimates and IQRs, B: posterior estimates and IQRs given failure data, C: posterior estimates and IQRs given failure data and maintenance data .

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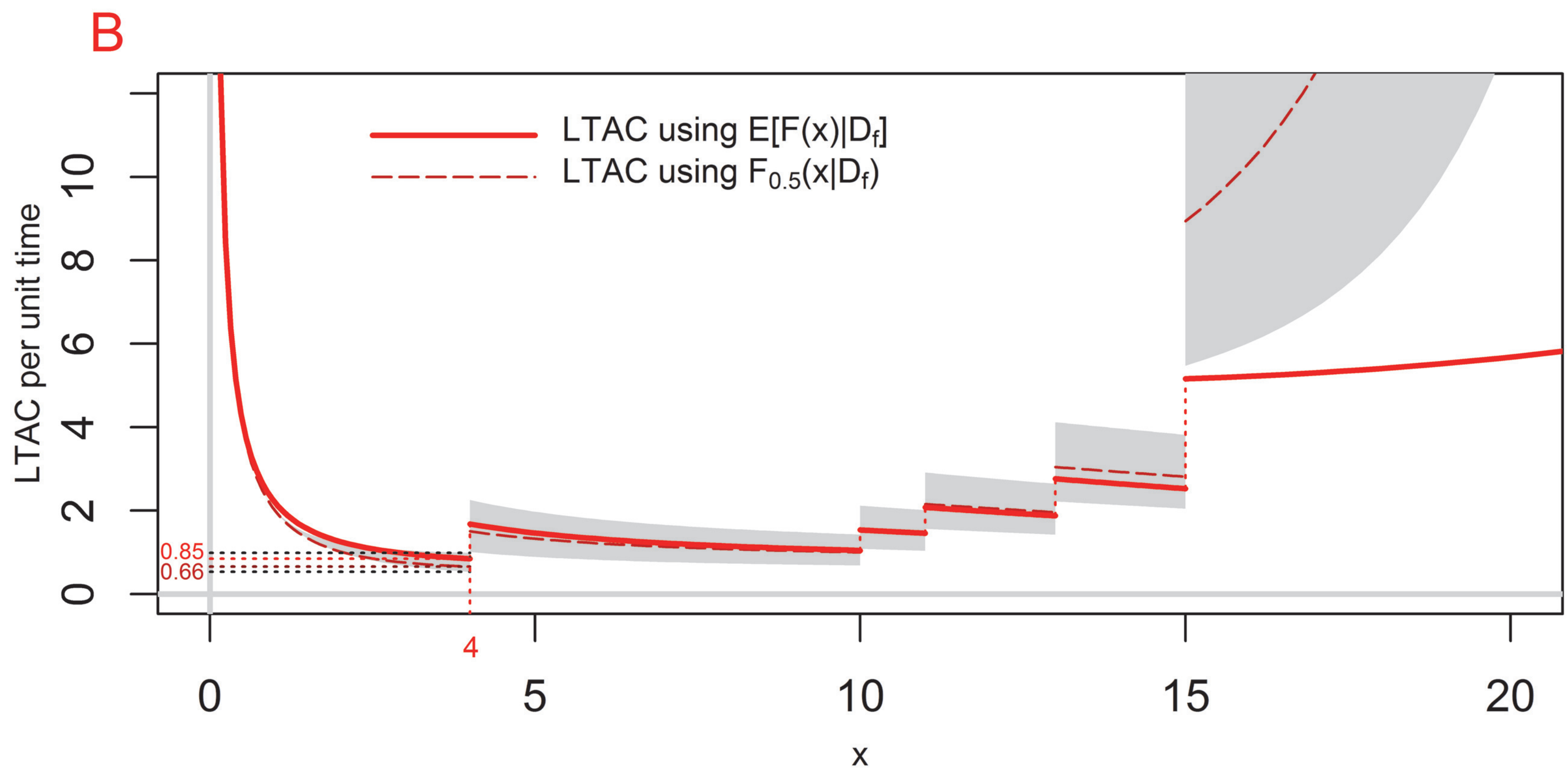

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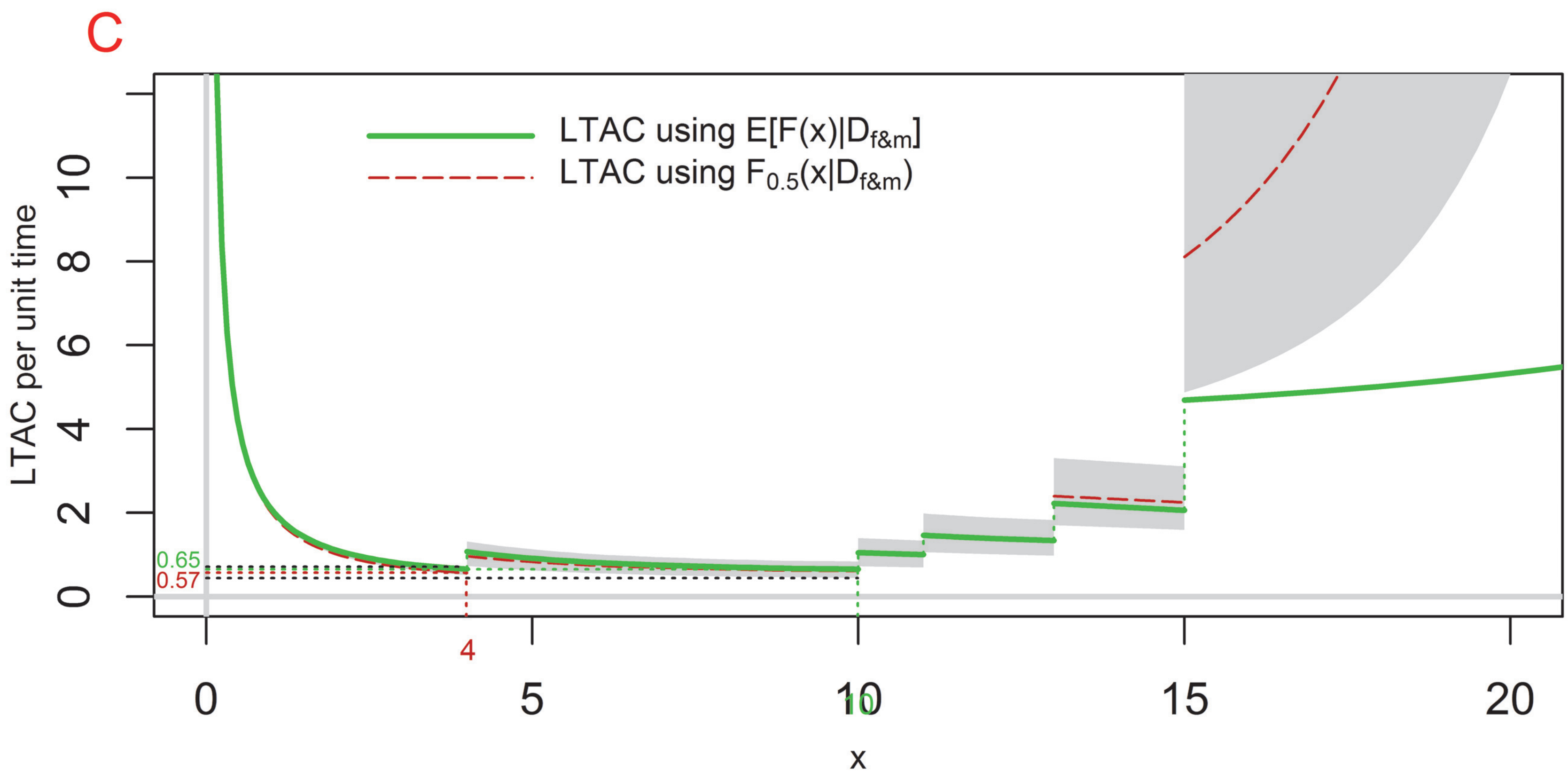

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Ferguson TS (1973). ***A Bayesian Analysis of some nonparametric problems,*** Annals of Statistics, Vol. 2, pp. 209-230.

Suzarla V and van Ryzin J (1976). Nonparametric Bayesian estimation of survival curves from incomplete observations. *Journal of the American Statistical Association*, 71 (356), 897-902.

Cooke RM (1991). ***Experts in Uncertainty: Opinion and Subjective Probability in Science, New York***, Oxford University Press.

Dekker R (1996). Application of maintenance optimization models: a review and analysis. *Reliability Engineering and System Safety*, 51, pp. 229-240.

Mazzuchi TA and Soyer R (1996). A Bayesian Perspective on Some Replacement Strategies. *Reliability Engineering and System Safety*, 51, 295-303.

Van Dorp JR and S. Kotz (2003). Generalized Trapezoidal Distributions. *Metrika*, 58 (1), 85-97.

Herrerías-Velasco JM, Herrerías-Pleguezuelo R and Van Dorp JR (2009). **The Generalized Two-Sided Power Distribution**, *Journal of Applied Statistics*, 36 (5), 573-587.

Morris DE, Oakley JE and JA Crowe (2014), **A web-based tool for eliciting probability distributions from experts**. *Environmental Modelling & Software*, 52, 1-4.