Short Notes

Functional and Topological Relations Among Banyan Multistage Networks of Differing Switch Sizes

Abdou Youssef and Bruce Arden

Abstract— **Equivalence relations among banyan multistage interconnection networks (MIN) have received much attention in recent years, mainly because two equivalent networks can simulate one another. Most of the efforts in this area have focused on comparing networks of the same building block (i.e., switch) size. This paper studies relations among MIN's of differing switch sizes. If two $N \times N$ networks $\mathcal{W}$ and $\mathcal{W}'$ have switch sizes r and s, respectively, and if $r > s$, then $\mathcal{W}$ realizes a larger number of permutations than $\mathcal{W}'$. Consequently, the two networks can never be equivalent. However, $\mathcal{W}$ may realize all the permutations of $\mathcal{W}'$, in which case $\mathcal{W}$ is said to functionally cover $\mathcal{W}'$ in the strict sense. More generally, $\mathcal{W}$ is said to functionally cover $\mathcal{W}'$ in the wide sense if the terminals of $\mathcal{W}$ can be relabeled so that $\mathcal{W}$ realizes all the permutations of $\mathcal{W}'$. In this paper, functional covering is topologically characterized, and an optimal algorithm to decide strict functional covering is developed.**

Index Terms— **Banyan multistage networks, digit permutations, network equivalence, realizable permutation, switch size.**

I. Introduction

Banyan multistage interconnection networks (MIN) have received much attention in recent years because of their key role in parallel processing systems [1]–[4], [9], [10]. These networks have the unique path property, that is, there is a unique path between every source and every destination. As a result, they are efficiently controllable but do not realize all permutations [6], except in the special case when the network is a single crossbar.

As different MIN's may realize different sets of permutations, several reserach efforts have been directed toward functional equivalence relations among these networks [7], [10], [11], and algorithms to decide functional equivalence have been devised [7], [13]. However, all the research efforts have focused on symmetric equivalence relations, and often among networks of the same switch size.

In this paper, nonsymmetric relations among MIN's of *differing* switch sizes are addressed. If two $N \times N$ MIN's have differing switch sizes, then the MIN with the larger switch size realizes a larger set of permutations. Therefore, the two networks can never be functionally equivalent. Consequently, the relation of inclusion of the smaller set in the larger set, or, stated otherwise, the relation of *functional covering* of one network by the other, is the appropriate relation to address. A network is said to *functionally cover* another network *in the strict sense* if the permutations realizable by the second network are realizable by the first network. A network is said to *functionally cover* another network *in the wide sense* if the terminals of the first network can be relabeled so that it functionally covers the second network in the strict sense.

To better see the merit of the covering problem, consider omega networks [6]. It may seem intuitive that if $s < r$, then an $N \times N$ omega network with $r \times r$ crossbar switches as building blocks realizes all the permutations of another $N \times N$ omega network with $s \times s$ crossbar switches as building blocks. However, this is not always the case. Take for example two 64×64 omega networks, Ω_1 and Ω_2, with 8×8 and 4×4 crossbar switches as building blocks, respectively. Ω_1 realizes $(8!)^{2\times 8} = (40320)^{16}$ permutations and Ω_2 realizes $(4!)^{3\times 16} = (13824)^{16}$ permutations, a much smaller set than that of Ω_1. However, not every permutation realizable by Ω_2 is realizable by Ω_1. In fact, using the characterization of nonconflicting source-destination paths in omega [6], it can be shown that the paths $0 \rightarrow 0$ and $24 \rightarrow 5$ do not conflict in Ω_2 but do conflict in Ω_1. Hence, there exists a permutation that maps 0 to 0 and 24 to 5, and that is realizable by Ω_2 but not by Ω_1.

This paper will investigate functional covering among MIN's of the same terminal size but of differing switch sizes. Necessary and sufficient conditions for a MIN to functionally cover another MIN will be determined. Specifically, it will be shown that an $N \times N$ MIN of $r \times r$ switches functionally covers in the strict sense another $N \times N$ MIN of $s \times s$ switches if and only if r is a power of s (i.e., $r = s^l$ for some integer l) and the topology of the second MIN can be derived from the first MIN by replacing each switch of the first MIN by some $r \times r$ MIN of $s \times s$ switches. Based on this topological characterization of functional covering, an optimal algorithm to decide functional covering in the strict sense will be given. Wide functional covering decision algorithms are harder to develop and are left for future work.

The paper will also address network covering in a special class of MIN's, called digit permutation networks (DPN) [5], [12]. In a digit permutation network, the inter-column interconnections are digit permutations that permute digits in a specified manner. Most existing MIN's are digit permutation networks. These include omega [6], omega inverse, the indirect binary n-cube [8], the generalized cube network [10], and the baseline network [11]. It was shown in [5] that all $N \times N$ digit permutation networks of the same switch size are widely functionally equivalent. It is also known that an $N \times N$ omega of switch size r strictly functionally covers any other $N \times N$ omega of switch size s if r is a power of s [6]. Combining the last two facts, it will be concluded that every $N \times N$ digit permutation network with $r \times r$ switches functionally covers in the wide sense any other $N \times N$ digit permutation network of $s \times s$ switches such that r is a power of s. This enables the first network to simulate the second by appropriately relabeling the terminals of the first network.

The paper is organized as follows. The next section will review multistage networks and rigorously define the functional covering relations and related concepts. Section III will establish the necessary and sufficient conditions in order for a MIN to functionally cover another MIN. The algorithm to decide strict functional covering will be developed in Section IV. The wide functional covering among digit permutation networks will be addressed in Section V. Section VI will give some concluding remarks and future directions.

Manuscript received August 7, 1990; revised August 12, 1991.

A. Youssef is with the Department of Electrical Engineering and Computer Science, The George Washington University, Washington, DC 20052.

B. Arden is with the College of Engineering and Applied Science, University of Rochester, Rochester, NY 14627.

IEEE Log Number 9205855.

II. Preliminaries and Fundamental Concepts

In this section MIN's are specified, functional and topological equivalence relations among them are reviewed, and functional and topological covering relations will be defined.

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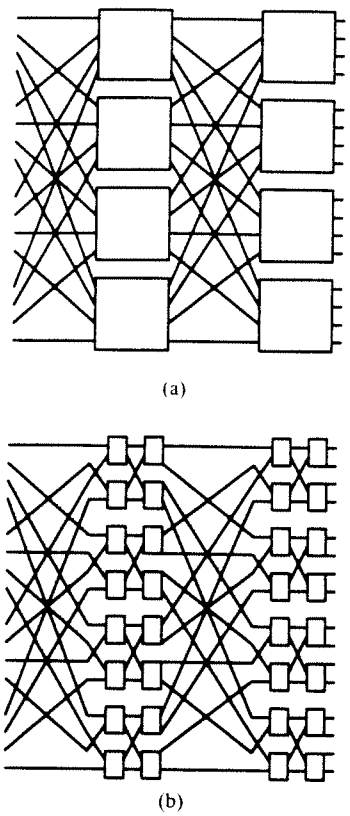

(a)

(b)

Fig. 1. Two banyan multistage networks.

An $N \times N$ MIN has N input terminals, N output terminals, and k interconnected columns of N/r $r \times r$ crossbar switches, where $N = r^k$ and $r \geq 2$. N is called the terminal size of the network. Each $r \times r$ crossbar switch realizes all $r!$ permutations. The interconnection between every two successive columns is a permutation of $S_N = \{0, 1, \cdots, N-1\}$. The connectivity of these networks is such that they have the unique path property. That is, between every input terminal i and every output terminal j there is one and only one path, which will be denoted $i \rightarrow j$. The class of these networks is denoted MIN(r, k). Fig. 1 shows two MIN's.

For ease of reference, the input (i.e., left) terminals of networks in MIN(r, k) are labeled $0, 1, \cdots, N-1$ from top to bottom, and so are the output (right) terminals. The columns are numbered $0, 1, \cdots, k-1$ from left to right, and the switches of each column are labeled $0, 1, \cdots, (N/r) - 1$ from top to bottom.

If $\mathcal{W}$ is a network of MIN(r, k) and f a permutation of S_N, f can be viewed as an interconnection and can be appended to the right end of $\mathcal{W}$, forming a network denoted $\mathcal{W}f$. Another way of viewing $\mathcal{W}f$ is as $\mathcal{W}$ except that the output terminals of $\mathcal{W}$ are relabeled by f, that is, output terminal j is relabeled $f(j)$, for every $j = 0, 1, \cdots, N-1$. Similarly, f can be appended to the left of $\mathcal{W}$ forming $f\mathcal{W}$. Viewed differently, $f\mathcal{W}$ is the same as $\mathcal{W}$ except that the input terminals of $\mathcal{W}$ are relabeled by f^{-1}, that is, every input terminal i of $\mathcal{W}$ is relabeled $f^{-1}(i)$.

The composition of functions is taken here from left to right, that is, $(x)fg = g(f(x))$. If $P(\mathcal{W})$ denotes the set of permutations realizable by $\mathcal{W}$, then $P(g\mathcal{W}f) = \{ghf \mid h \in P(\mathcal{W})\}$, which clearly follows from the definition of $g\mathcal{W}f$ and the left-to-right view of composition.

Two networks $\mathcal{W}$ and $\mathcal{W}'$ in MIN(r, k) are *strictly functionally equivalent* if they realize the same permutations. The two networks are *widely functionally equivalent* if there exist two permutations g and f of S_N such that $g\mathcal{W}f$ and $\mathcal{W}'$ are strictly functionally equivalent. A network $\mathcal{W}$ in MIN(r, k) is said to *functionally cover* another network $\mathcal{W}'$ in MIN(s, k') *in the strict sense* if $P(\mathcal{W}')$ is a subset of $P(\mathcal{W})$. The network $\mathcal{W}$ is said to functionally cover $\mathcal{W}'$ *in the wide sense* if there exist two permutations g and f of S_N such that $g\mathcal{W}f$ realizes all the permutations of $\mathcal{W}'$.

Topological relations are defined next. To this effect, denote by $\mathcal{G}(\mathcal{W})$ the standard directed graph abstraction for any MIN $\mathcal{W}$. The nodes of $\mathcal{G}(\mathcal{W})$ are the switches as well as the input/output terminals of $\mathcal{W}$. The edges of $\mathcal{G}(\mathcal{W})$ are the links of $\mathcal{W}$ directed rightward. Two networks $\mathcal{W}$ and $\mathcal{W}'$ in MIN(r, k) are said to be *widely topologically equivalent* if $\mathcal{G}(\mathcal{W})$ and $\mathcal{G}(\mathcal{W}')$ are isomorphic graphs. $\mathcal{W}$ and $\mathcal{W}'$ are *strictly topologically equivalent* if the graph isomorphism between $\mathcal{G}(\mathcal{W})$ and $\mathcal{G}(\mathcal{W}')$ corresponds every input (resp., output) terminal of $\mathcal{W}$ to the same input (resp., output) terminal of $\mathcal{W}'$. It is clear that if two networks are strictly (resp., widely) topologically equivalent, then they must be strictly (resp., widely) functionally equivalent.

An $N \times N$ network $\mathcal{W}$ in MIN(r, k) is said to *topologically cover* another $N \times N$ network $\mathcal{W}'$ in MIN(s, k') *in the strict sense* if r is a power of s (i.e., $r = s^l$ for some integer l) and the switches of $\mathcal{W}$ can be replaced by networks in MIN(s, l) such that the resulting network is strictly topologically equivalent to $\mathcal{W}'$. The network in Fig. 1(a) topologically covers the network in Fig. 1(b). The network $\mathcal{W}$ is said to topologically cover $\mathcal{W}'$ *in the wide sense* if $\mathcal{W}$ can be made to topologically cover $\mathcal{W}'$ in the strict sense by relabeling the input and/or output terminals of $\mathcal{W}$. As the permutations realizable by any network in MIN(s, l) form a subset of the $r!$ permutations realizable by an $r \times r$ switch, it follows that strict topological covering implies strict functional covering. Similarly, wide topological covering implies wide functional covering.

Since the proof of the equivalence between functional covering and topological covering will proceed by induction on the number of columns of the covering network, the class of *incomplete* MIN's will be introduced. Define by IMIN(r, k, t) the class of $N \times N$ networks of t columns of $r \times r$ switches, where $N = r^k$, $t \leq k$ and each input terminal can reach exactly r^t output terminals through unique paths but cannot reach any of the other output terminals. Clearly, IMIN(r, k, k) = MIN(r, k). All the functional and topological relations can be extended to networks in IMIN's with only one addition to strict functional covering as follows. A network $\mathcal{W}$ in IMIN(r, k, t) *functionally covers* another network $\mathcal{W}'$ in IMIN(s, k', t') *in the strict sense* if $P(\mathcal{W}')$ is a subset of $P(\mathcal{W})$ and the set of output terminals reached from an input terminal in $\mathcal{W}'$ is the same set reached in $\mathcal{W}$ from the same input terminal. This second condition is equivalent to saying that $r^t = s^{t'}$, and is hence superfluous in the case of MIN's, that is, when $t = k$ and $t' = k'$.

III. Equivalence Between Functional and Topological Covering

In this section it will be shown that a network in MIN(r, k) functionally covers another network in MIN(s, k') in the strict sense (resp., wide sense) if and only if the first network topologically covers the second network in the strict sense (resp., wide sense). The equivalence in the strict sense is shown first, and the equivalence in the wide sense will follow as a corollary.

The proof will proceed by induction on the number of columns. For that reason, the equivalence between strict functional covering and strict topological covering will be established for the IMIN classes. A number of supporting lemmas are shown first.

Lemma 1: Let $\mathcal{W}$ be in IMIN(r, k, l), E_i a subset of switches of some column i of $\mathcal{W}$, and $E_{i+1}, \cdots, E_j$ defined inductively as follows. E_t is the set of switches in column t that are linked to

switches in E_{t-1}, for $t = i+1, i+2, \cdots, j$. Then

i) $|E_i| \le |E_{i+1}| \le \cdots \le |E_j|$.

ii) If $|E_i| \ge |E_j|$, then $|E_i| = |E_{i+1}| = \cdots = |E_j|$ and the switches of $E_i, E_{i+1}, \cdots, E_j$ along with their interconnections form an independent $n \times n$ subnetwork of $\mathcal{W}$, where $n = r|E_i|$.

Proof: i) The number of links going out of the switches of E_i is $r|E_i|$. All these links come into the switches of E_{i+1}. As the total number of links incoming to the switches of E_{i+1} is $r|E_{i+1}|$, it follows that $r|E_i| \le r|E_{i+1}|$, implying that $|E_i| \le |E_{i+1}|$. The remaining inequalities can be shown similarly.

ii) If $|E_i| \ge |E_j|$, then, using i), we have $|E_i| = |E_{i+1}| = \cdots = |E_j|$. This shows that all the links coming to the switches of E_{i+1} are from the switches of E_i, those coming into the switches of E_{i+2} are from E_{i+1}, and so on. As a result, the switches of $E_i, E_{i+1}, \cdots, E_j$ along with their interconnections form an $n \times n$ subnetwork of $\mathcal{W}$, where $n = r|E_i|$. □

Lemma 2: Let $\mathcal{W}$ be in IMIN(r, k, t) and $\mathcal{W}'$ in IMIN(s, k', t') such that $\mathcal{W}$ functionally covers $\mathcal{W}'$ in the strict sense. Let also $a_1, a_2, \cdots, a_r$ be the input terminals linked to switch x in column 0 of $\mathcal{W}$, and j_0 the label of the leftmost column in $\mathcal{W}'$ in which there exists at least one switch u that is reachable from all $a_1, a_2, \cdots, a_r$ in $\mathcal{W}'$. Then $(k'/k) - 1 \le j_0 < k'/k$ and hence $j_0 = \lceil (k'/k) - 1 \rceil$.

Proof: To show that $(k'/k) - 1 \le j_0$, note that the total number of input terminals reachable from u in $\mathcal{W}'$ is s^{j_0+1}. Thus, $s^{j_0+1} \ge r = s^{k'/k}$ and, therefore, $j_0 \ge (k'/k) - 1$.

To show that $j_0 < k'/k$, we will reason by contradiction, assuming that $j_0 \ge k'/k$. Let B_i be the set of output terminals of $\mathcal{W}'$ reachable from output port i of switch u, for $i = 1, 2, \cdots, s$. Let $A_1, A_2, \cdots, A_r$ be the sets of output terminals of $\mathcal{W}$ that are reachable from output ports $1, 2, \cdots, r$ of switch x, respectively, as shown in Fig. 2. Let also $O_u = \cup_{i=1}^{s} B_i$ be the set of all the output terminals of $\mathcal{W}'$ reachable from u, $I = \{j \mid O_u \cap A_j \ne \emptyset\}$, and $E = \cup_{j \in I} A_j$. We clearly have $O_u \subseteq E$. We will derive next a contradiction in each case whether $O_u = E$ or O_u is proper subset of E. The approach is to find two paths that conflict in $\mathcal{W}$ but not in $\mathcal{W}'$, contradicting that $\mathcal{W}$ functionally covers $\mathcal{W}'$ in the strict sense.

Case 1: $O_u = E$. Then for every $j \in I$, $A_j \subseteq \cup_{i=1}^{s} B_i$. We will show that $|B_i| < |A_j|$. To see this, note that $|B_i| = s^{t'-j_0-1} \le s^{t'-(k'/k)-1}$. Since $\mathcal{W}$ functionally covers $\mathcal{W}'$, it follows that $r^t = s^{t'}$ and $r^k = s^{k'}$. Therefore, $k'/k = t'/t$, $s = r^{t/t'}$ and hence $s^{t'-(k'/k)-1} = r^{(t/t')(t'-(t'/t)-1)} = r^{t-1-(t/t')} < r^{t-1}$. As $|A_j| = r^{t-1}$, it follows that $|B_i| < |A_j|$. Therefore, A_j cannot be contained in any single B_i. Consequently, there exist i_1 and i_2 such that $A_j \cap B_{i_1} \ne \emptyset$ and $A_j \cap B_{i_2} \ne \emptyset$. Let h be in $A_j \cap B_{i_1}$ and h' in $A_j \cap B_{i_2}$. Take any input terminal a_l such that the paths from a_1 to u and from a_l to u do not conflict in $\mathcal{W}'$ (a_l must clearly exist). It can now be seen that the paths $a_1 \to h$ and $a_l \to h'$ do not conflict in $\mathcal{W}'$ because $B_{i_1} \cap B_{i_2} = \emptyset$, while these same paths conflict in $\mathcal{W}$ because both h and h' are in A_j and the two paths have to go through the output port j of switch x. This contradicts the fact that $\mathcal{W}$ functionally covers $\mathcal{W}'$ in the strict sense.

Case 2: O_u is a proper subset of E. Let $O_u^c = E - O_u$ which is nonempty. As $O_u^c \subseteq E$ and $O_u^c \ne \emptyset$, there exists $i_0 \in I$ such that $O_u^c \cap A_{i_0} \ne \emptyset$. Since i_0 is in I, we have $O_u \cap A_{i_0} \ne \emptyset$. Therefore, there exist two output terminals h and h' such that $h \in O_u \cap A_{i_0}$ and $h' \in O_u^c \cap A_{i_0}$. Here too the paths $a_1 \to h$ and $a_l \to h'$ do not conflict in $\mathcal{W}'$ while these same paths conflict in $\mathcal{W}$ because both h and h' are in A_{i_0} and the two paths have to go through the switch x. Thus, we have the same contradiction as in the previous case.

Therefore, the assumption $j_0 \ge k'/k$ must be false. □

Lemma 3: Let $\mathcal{W}$, $\mathcal{W}'$, $a_1, a_2, \cdots, a_r$, x, j_0 and u be as in the previous lemma. Then the following statements hold:

i) Let F be the set of switches in column j_0 of $\mathcal{W}'$ that are reachable from every input terminal in the set $\{a_1, a_2, \cdots, a_r\}$, and F_i the set of switches in column j_0 that are reachable from input terminal a_i. Then $F_i = F$ for every $i = 1, 2, \cdots, r$.

ii) For every two output ports c and d of the switches in F there exist two input terminals a_{j_1} and a_{j_2} such that the two paths $a_{j_1} \to c$ and $a_{j_2} \to d$ going through the first $j_0 + 1$ columns of $\mathcal{W}'$ do not conflict.

iii) $r = s^{j_0+1}$.

iv) The switches that can be traced from $a_1, a_2, \cdots, a_r$ rightward to column j_0 in $\mathcal{W}'$ form an $r \times r$ independent subnetwork S in MIN$(s, j_0 + 1)$.

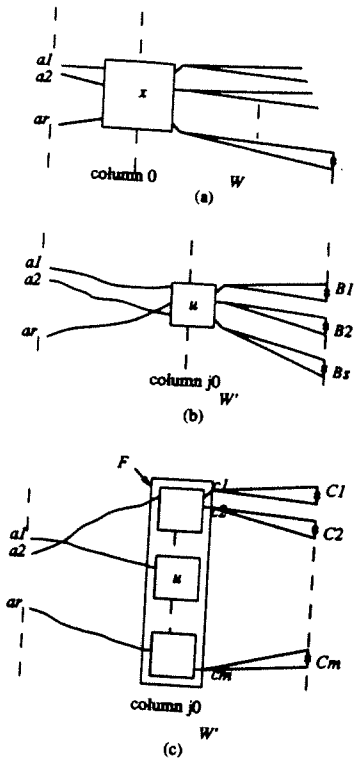

Fig. 2. Illustration of Lemma 2.

Proof: i) Clearly F is a subset of F_i. To show that F_i is a subset of F, let p be an arbitrary switch in F_i. Let also d be an output terminal reachable from a_i through p in $\mathcal{W}'$. Since $\mathcal{W}$ functionally covers $\mathcal{W}'$ in the strict sense, it follows that the path $a_i \to d$ is realizable in $\mathcal{W}$. This path must go through switch x, and thus, $a_1, a_2, \cdots, a_r$ can all reach d in $\mathcal{W}$. Due to the strict functional covering, d must be reachable from $a_1, a_2, \cdots, a_r$ in $\mathcal{W}'$ as well. There must then exist a switch that is reachable from $a_1, a_2, \cdots, a_r$ in $\mathcal{W}'$ on their way to d. Let v be the leftmost such switch and j_1 the label of the column of v. Lemma 3 applies to v and j_1 as it does to u and j_0. Therefore, $j_1 = \lceil (k'/k) - 1 \rceil$. Hence, $j_1 = j_0$ and v is in column j_0. As the path $a_i \to d$ goes through switch p and switch v in $\mathcal{W}'$, and as p and v are in the same column, we must have $p = v$. Consequently, p is reachable from $a_1, a_2, \cdots, a_r$ in $\mathcal{W}'$, implying that p belongs to F. It follows that $F_i = F$ for every $i = 1, 2, \cdots, r$.

ii) We have two cases. The first is when both c and d are two output ports of one switch. The second is when they are output ports of two distinct switches.

Case 1: Let v be the switch of which c and d are two output terminals. Let G_i be the set of input terminals of $\mathcal{W}'$ that are reachable from input port i of v, for $i = 1, 2, \cdots, s$. As $v \in F$, the input terminals $a_1, a_2, \cdots, a_r$ are reachable from v in $\mathcal{W}'$. Therefore,

$$\{a_1, a_2, \cdots, a_r\} \subseteq \cup_{i=1}^{s} G_i. \tag{1}$$

It is also clear that

$$|G_i| = s^{j_0} < s^{\frac{k'}{k}} = r. \tag{2}$$

It follows from (1) and (2) that the set $\{a_1, a_2, \cdots, a_r\}$ cannot be contained in a single G_i. Therefore, there exist two distinct integers i_1 and i_2, and two input terminals a_{j_1} and a_{j_2} such that $a_{j_1} \in G_{i_1}$ and $a_{j_2} \in G_{i_2}$. Consequently, the paths $a_{j_1} \to c$ and $a_{j_2} \to d$ do not conflict in $\mathcal{W}'$.

Case 2: Let v and z be the two switches such that c is an output port of v and d an output port of z. Let G_{i_1}, G_{i_2}, a_{j_1} and a_{j_2} be as in the previous case, and H_i the set of input terminals of $\mathcal{W}'$ that are reachable from input port i of switch z. Here too the a_i's belong to the union of the H_i's. Therefore, a_{j_2} must belong to some H_{i_3}. The paths $a_{j_1} \to i_1$ and $a_{j_2} \to i_3$ do not conflict in $\mathcal{W}'$ because if they did, then a_{j_2} would be reachable from input port i_1 of v and hence $G_{i_1} \cap G_{i_2} \neq \emptyset$, which is impossible because the sets of input terminals reachable from two distinct input ports of a switch are disjoint. Consequently, the paths $a_{j_1} \to c$ and $a_{j_2} \to d$ do not conflict in $\mathcal{W}'$.

iii) Let $m = |F| = |F_i| = s^{j_0+1}$, and $c_1, c_2, \cdots, c_m$ be the output ports of the switches in F (also in F_i). Let also C_i be the set of output terminals reachable from c_i in $\mathcal{W}'$, and A_j be the set of output terminals reachable from output port j of switch x in $\mathcal{W}$. We will show that for every $j = 1, 2, \cdots, r$, there exists $i = 1, 2, \cdots, m$ such that $A_j = C_i$. Afterwards, noting that $|A_j| = r^{t-1}$, $|C_i| = s^{t'-j_0-1} = s^{t'}/s^{j_0+1} = r^t/s^{j_0+1}$, and $|C_i| = |A_j|$, we will conclude that $r = s^{j_0+1}$.

As all the switches in column j_0 reachable from a_i in $\mathcal{W}'$ form the set $F_i = F$, it follows that the output ports in column j_0 that are reachable from a_i are $c_1, c_2, \cdots, c_m$, and, consequently, the set of output terminals reachable from a_i in $\mathcal{W}'$ is $\cup_{i=1}^m C_i$. On the other hand, the set of output terminals reachable from input terminal a_i in $\mathcal{W}$ is $\cup_{j=1}^r A_j$. As $\mathcal{W}$ functionally covers $\mathcal{W}'$ in the strict sense, these two sets must be equal. Therefore, for every $j = 1, 2, \cdots, r$, A_j is a subset of $\cup_{i=1}^m C_i$. We claim that there must exist an i such that $A_j \subseteq C_i$, for otherwise, there would exist two distinct integers i_1 and i_2 such that $A_j \cap C_{i_1} \neq \emptyset$ and $A_j \cap C_{i_2} \neq \emptyset$. It would follow that there exist two output terminal labels $e \in A_j \cap C_{i_1}$ and $f \in A_j \cap C_{i_2}$. Using ii), we conclude that there exist a_{j_1} and a_{j_2} such that the paths $a_{j_1} \to c_{i_1}$ and $a_{j_2} \to c_{i_2}$ do not conflict in $\mathcal{W}'$. Therefore, the paths $a_{j_1} \to e$ and $a_{j_2} \to f$, going through c_{i_1} and c_{i_2}, respectively, do not conflict in $\mathcal{W}'$ because $C_{i_1} \cap C_{i_2} = \emptyset$. However, these same paths conflict in $\mathcal{W}$ over output port j of switch x because both e and f are in A_j. Consequently, there must exist i such that $A_j \subseteq C_i$. Noting that $|C_i| = r^t/s^{j_0+1} \leq r^{t-1} = |A_j|$, it follows that $A_j = C_i$. Using now the equality between the cardinalities between A_j and C_i as indicated earlier, we conclude that $r = s^{j_0+1}$.

iv) Let E_0 be the set of switches of column 0 of $\mathcal{W}'$ such that each switch is linked to at least one of $a_1, a_2, \cdots, a_r$. Let also E_i be the set of switches in column i that are linked to switches in E_{i-1}, for $i = 1, 2, \cdots, j_0$. After Lemma 3-i) we have

$$|E_0| \leq |E_1| \leq \cdots \leq |E_{j_0}|. \tag{3}$$

Every switch of E_{j_0} is clearly reachable from one of the inputs $a_1, a_2, \cdots, a_r$. Consequently, $E_{j_0} \subseteq \cup_{i=1}^r F_i = F$, and hence

$$|E_{j_0}| \leq |F_i| = s^{j_0}. \tag{4}$$

As the number of all input terminals linked to the switches of E_0 is equal to $s|E_0|$ on the one hand and $\geq |\{a_1, a_2, \cdots a_r\}| = r$, it follows that $|E_0| \geq r/s = s^{j_0+1}/s = s^{j_0} \geq |E_{j_0}|$. Thus,

$$|E_0| \geq |E_{j_0}|. \tag{5}$$

The inequalities of (3) and (5) enable us to use Lemma 3-ii) concluding that the switches of $E_0, E_1, \cdots, E_{j_0}$ along with their interconnections form an independent $n \times n$ network of $\mathcal{W}'$, where $n = s|E_0| = s^{j_0-1} = r$. As the total number of input terminals linked to the switches of E_0 is r, it follows that these input terminals are $a_1, a_2, \cdots, a_r$, and the lemma follows. □

Lemma 4: Let $\mathcal{W}$ be in IMIN(r, k, t) and $\mathcal{W}'$ in IMIN(s, k', t') such that $\mathcal{W}$ functionally covers $\mathcal{W}'$ in the strict sense. Let also $l = \log_s r$. The leftmost l columns of $\mathcal{W}'$ can be regrouped into $r \times r$ subnetworks in MIN(s, l). Furthermore, the regrouping can be done so that the leftmost interconnections of $\mathcal{W}$ and $\mathcal{W}'$ (between the input terminals and the leftmost column) become identical, and for every j and i, the set of output terminals reachable from output port i of switch j of column 0 of $\mathcal{W}$ is identical to the set of output terminals reachable from output terminal i of the jth subnetwork in $\mathcal{W}'$ (see Fig. 3).

Proof: After part iii) of the previous lemma, r is a power of s and hence l is an integer. After part iv) of that same lemma, every $r \times r$ switch u of column 0 of $\mathcal{W}$ corresponds to an $r \times r$ independent subnetwork in the l leftmost columns of $\mathcal{W}'$ with the same input terminals as those connected to switch u. Therefore, the l leftmost columns of $\mathcal{W}'$ can be regrouped so that the $r \times r$ subnetwork corresponding to switch j of column 0 of $\mathcal{W}$ is identifiable, positioned in the jth position (from the top), and having its inputs connected to the r input terminals in the same way as the corresponding input ports of switch j are connected to the same input terminals.

Following the same line of reasoning as in the proof of part iii) of Lemma 2, we conclude that for every output port i of switch j of column 0 of $\mathcal{W}$, there exists an output terminal p_i of the corresponding network such that the set of the labels of the output terminals of $\mathcal{W}$ reachable from i is the same as the set of output terminals of $\mathcal{W}'$ reachable from p. Accordingly, for every i, the output p_i of the subnetwork is relabeled i in order for the switch-subnetwork correspondence to be perfect. □

Now we are in a position to show by induction the equivalence between strict functional covering and strict topological covering.

Theorem 1: A network $\mathcal{W}$ in IMIN(r, k, t) functionally covers a network $\mathcal{W}'$ in IMIN(s, k', t') in the strict sense if and only if r is a power of s and $\mathcal{W}$ topologically covers $\mathcal{W}'$ in the strict sense.

Proof: We have seen before that topological covering implies functional covering. It remains to show the converse. Let $\mathcal{W}$ be a network in IMIN(r, k, t) that functionally covers another network $\mathcal{W}'$ in IMIN(s, k', t') in the strict sense. After Lemma 3-iii), r is power of s, that is, there exists a positive integer l such that $r = s^l$. It will be shown by induction on t that $\mathcal{W}$ topologically covers $\mathcal{W}'$ in the strict sense.

Basis step: $t = 1$, that is, $\mathcal{W}$ has only one column of switches. As $r^t = s^{t'}$, we have $r = s^{t'}$ and thus $l = t'$. After Lemma 2, the l leftmost columns of $\mathcal{W}'$, which in this case are all the columns of $\mathcal{W}'$, can be regrouped into $r \times r$ subnetworks in MIN(s, l) in perfect one-to-one correspondence with the switches of column 0 of $\mathcal{W}$. It can be easily seen that by replacing each subnetwork of $\mathcal{W}'$ by an $r \times r$ switch, we get a network identical to $\mathcal{W}$.

Induction step: Assume that if a network in IMIN$(r, k, t-1)$ functionally covers another network in IMIN(s, k', t') in the strict sense, the former network then topologically covers the latter in the strict sense. After the previous lemma, the l leftmost columns of $\mathcal{W}'$ can be regrouped into $r \times r$ subnetworks in perfect one-to-one correspondence with the switches of the leftmost column of $\mathcal{W}$ such that the leftmost interconnections of both networks become identical. Do the regrouping as described in the previous lemma, then delete the leftmost interconnection and these resulting subnetworks from $\mathcal{W}'$, and call the resulting network R' [Fig. 3(b)]. Delete also the

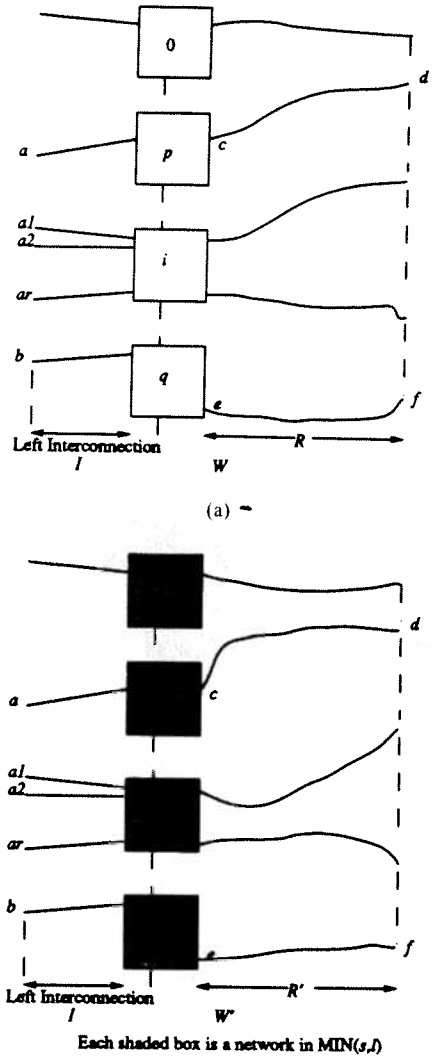

Fig. 3. Regrouping of switches in $\mathcal{W}'$.

leftmost interconnection and the leftmost column from $\mathcal{W}$, and call the resulting network R [Fig. 3(a)]. It can be easily established that R functionally covers R' in the strict sense by showing that every two source-destination paths that do not conflict in R' do not conflict in R. Using the inductive hypothesis, we conclude that R topologically covers R' in the strict sense. As the first stage of $\mathcal{W}$ topologically covers the portion of $\mathcal{W}'$ that is missing from R', it follows that $\mathcal{W}$ topologically covers $\mathcal{W}'$ in the strict sense. □

Theorem 2: A network $\mathcal{W}$ in IMIN(r, k, t) functionally covers a network $\mathcal{W}'$ in IMIN(s, k', t') in the wide sense if $\mathcal{W}$ topologically covers $\mathcal{W}'$ in the wide sense.

Proof: By definition, $\mathcal{W}$ topologically covers $\mathcal{W}'$ in the wide sense if and only if $\mathcal{W}$ topologically covers $\mathcal{W}'$ in the strict sense after appropriate relabeling of the terminals of $\mathcal{W}'$, that is, by Theorem 1, if and only if $\mathcal{W}$ functionally covers $\mathcal{W}'$ in the strict sense after appropriate relabeling of the terminals of $\mathcal{W}'$. As the latter statement is equivalent to saying that $\mathcal{W}$ functionally covers $\mathcal{W}'$ in the wide sense (by definition), the theorem follows. □

IV. Network Covering Algorithm

This section outlines an algorithm NETWORK-COVER that takes two $N \times N$ networks $\mathcal{W}$ in MIN(r, k) and $\mathcal{W}'$ in MIN(s, k') as input and decides if $\mathcal{W}$ functionally covers $\mathcal{W}'$ in the strict sense. The algorithm is a refinement of the effective proof of Theorem 3. The algorithm makes use of the equivalence between functional covering and topological covering, and also of the strict functional/topological equivalence algorithm EQUIVALNCE of [13].

NETWORK-COVER($\mathcal{W}, \mathcal{W}'$)

1. Construct the graphs $\mathcal{G}(\mathcal{W})$ and $\mathcal{G}(\mathcal{W}')$.
2. Let $l = \log_s r$. If l is not an integer, then report that $\mathcal{W}$ does not strictly cover $\mathcal{W}'$ and stop.
3. View the columns of switch-nodes of $\mathcal{G}(\mathcal{W}')$ as grouped into k consecutive bands, where the ith band consists of columns $i \times l, i \times l + 1, i \times l + 2 \cdots, (i+1) \times l - 1$.
4. For each band, do a column-wise sweep of the nodes from left to right to check if the band consists of N/r connected components and if each component is the abstract graph of some network in MIN(s, l) (i.e., the component is a multilayer graph of l columns of s nodes each). If the test fails, report that $\mathcal{W}$ does not strictly cover $\mathcal{W}'$ and stop. If the test succeeds, replace each component by a single node and denote by $\mathcal{G}''$ the resulting graph. Note that the inter-column edges in $\mathcal{G}''$ are the edges between the corresponding bands in $\mathcal{G}(\mathcal{W}')$.
5. Call the EQUIVALENCE($\mathcal{G}(\mathcal{W}), \mathcal{G}''$) algorithm of [13]. If $\mathcal{G}(\mathcal{W})$ and $\mathcal{G}''$ are strictly topologically equivalent, report that $\mathcal{W}$ strictly covers $\mathcal{W}'$; otherwise, report that $\mathcal{W}'$ does not strictly cover $\mathcal{W}'$.

Time complexity: Step 1 and step 4 take time proportional to the size of $\mathcal{W}'$, which is $O(N \log_s N)$. Step 5, which is the call to EQUIVALENCE, takes $O(N \log_r N)$ [13]. Steps 2 and 3 are trivial. Thus, the algorithm takes $O(N \log_s N)$, which is clearly optimal up to a constant factor.

V. Covering Relations Among Digit Permutation Networks

One common feature in the definitions of most existing MIN's is that the interconnections between columns are bit permutations. The well-known shuffle interconnection is an example. Among the reasons for using these permutations as interconnections are their regularity, rich structure and ease of analysis. Therefore, the MIN's that have as inter-column interconnections bit permutations or, in the general case where the switches are $r \times r$, digit permutations that permute digits of r-ary labels, are of special interest.

Formally speaking, a permutation f of $S_N = \{0, 1, \cdots, N-1\}$, where $N = r^k$, is a *digit permutation* in the system of base r if there exists a permutation π of $S_k = \{0, 1, \cdots, k-1\}$ such that $f(x_{k-1} \cdots x_1 x_0) = x_{\pi(k-1)} \cdots x_{\pi(1)} x_{\pi(0)}$, where $x_{k-1} \cdots x_1 x_0$ is an arbitrary k-digit r-ary label. An $N \times N$ *digit permutation network* (DPN) in MIN(r, k) is a MIN whose interconnections are digit permutations of S_N in the system of base r.

Digit permutation networks in MIN(r, k) have been shown by Kruskal and Snir [5] to be widely equivalent to one another, and in particular, to the omega network $\Omega(r, k)$. Using the characterization of Ω-realizable permutations derived by Lawrie [6], it follows that $\Omega(r, k)$ strictly covers $\Omega(s, k')$ if (and only if) r is a perfect power of s. Consequently, by simple transitivity, we conclude that every DPN $\mathcal{W}$ in MIN(r, k) widely covers every DPN $\mathcal{W}'$ in MIN(s, k') if and only if r is a perfect power of s. In order for $\mathcal{W}$ to simulate $\mathcal{W}'$ in the case r is a power of s, the input output terminals of $\mathcal{W}$ need to be relabeled somehow. An $O(N \log_s N)$ relabeling algorithm was developed in [12].

VI. Conclusions

In this paper we studied functional covering and topological covering relations and the equivalence between these two types of

relations among banyan multistage interconnection networks. The equivalence between functional covering and topological covering was established. Based on this equivalence, an optimal algorithm to decide strict functional covering was given. Finally, it was shown that every digit permutation networks functionally covers in the wide sense every other digit pertmutation network of the same terminal size if the switch size of the former network is a perfect power of the switch size of the latter network.

Future work includes 1) the development of efficient wide covering algorithms and 2) the study of functional and topological relations among networks that differ in terminal size as well as in switch size.

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