A Parallel Algorithm for Random Walk Construction with 
Application to the Monte Carlo Solution of Partial Differential Equations

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Abstract

Random walks are widely applicable in statistical and scientific computations. In particular, they are used in the Monte Carlo Method to solve elliptic and parabolic partial differential equations (PDE's). This method holds several advantages over other methods for (PDE's) as it solves problems with irregular boundaries and/or discontinuities, gives solutions at individual points, and exhibits great parallelism. However, the generation of each random walk in the Monte Carlo method has been done sequentially because each point in the walk is derived from the preceding point by moving one grid step along a randomly selected direction. In this paper, we present a parallel algorithm for the random walk generation in regular as well as irregular regions. The algorithm is based on parallel prefix computations, and takes \( O(L \log n) \) time, where \( L \) is the length of the random walk, and \( n \) is the number of processors. The communication structure of the algorithm is shown to ideally fit on a hypercube of \( n \) nodes.

§1. Introduction

The Monte Carlo Method has been studied and used to solve elliptic and parabolic partial differential equations (PDE) [5], [6], [11]. It holds several advantages over other methods, such as solving problems with irregular boundaries and/or discontinuities; giving solutions at single points independently from the solutions at other points; and allowing for great parallelism.

The great amount of parallelism is drawn from the fact that the solution at different points are independent, paving the way to independent processes that can be executed in parallel. Moreover, the solution at each point consists of evaluating a "primary estimator" (to be defined later) along a large number of random walks, then averaging these values. The random walks are independent and
therefore constructable in parallel, and the estimations along the random walks can be computed in parallel as well. Such parallel algorithms have been studied [1], and various computer architectures for their execution have been proposed [2], [3], [12].

A much less obvious amount of parallelism can be introduced into the evaluation of the primary estimator along a random walk. It is less obvious because the random walk is constructed sequentially making the computation proportional to the length of the random walk.

In this paper we develop and study a parallel algorithm for the construction of random walks and along with it the evaluation of the primary estimator, reducing the time of this part of the solution from $O(L)$ to $O(\log L)$, where $L$ is the length of the random walk. It should be noted that this parallel random walk generation algorithm has other applications in statistical and scientific computations where random walks and monte carlo methods are used.

This paper is organized as follows. The next section presents briefly the Monte Carlo Method for PDE's and points out all the possible areas of parallelism. Section 3 introduces the intra-walk parallelism and gives a parallel algorithm for the random walk computation. Conclusions are presented in section 4.

§2. The Monte Carlo Method and its Inherent Parallelism

Let

$$AU_{zz} + 2BU_{zy} + CU_{yy} + DU_x + EU_y + F = 0$$

(1)

be a PDE and $\Delta$ a region with boundary $\gamma$. The factors $A$, $B$, $C$, $D$, $E$ and $F$ and the unknown function $U$ are functions of $x$, $y$ and possibly the time variable $t$.

The Monte Carlo Method is used to solve the following two problems:

A. The Elliptic PDE Problem:

$U$, $A$, $B$, $C$, $D$, $E$ and $F$ are time-independent and $B^2 - AC < 0$ on $\Delta$. The problem is solve equation (1) subject to the boundary condition:

$$U(x,y) = \phi(x,y) \text{ if } (x,y) \in \gamma$$

(2)

B. The parabolic PDE problem:
$U, A, B, C, D, E$ and $F$ are time-dependent and $B^2 - AC = 0$ on $\delta$. The problem is solve
equation (1) subject to:

**Boundary Condition:** $U(x, y, t) = \phi(x, y, t)$ if $(x, y) \in \gamma$  \hspace{1cm} (3)

**Initial Condition:** $U(x, y, 0) = g(x, y)$ if $(x, y) \in \Delta$  \hspace{1cm} (4)

The region $\Delta$ is divided into a regular grid of size $h$. Each point $P_0$ of the grid (except the boundary
points) has 5 neighbors $P_1, P_2, \ldots, P_5$ as depicted in Figure 1. We denote by $\delta_i$ the direction
along which we move from $P_0$ to $P_i$, where $i = 1, 2, 3, 4, 5$. A random number generator (RNG)
generates random directions (i.e., $\delta_1, \delta_2, \ldots, \delta_5$). A random walk starting at $P_0$ is constructed by
moving away from $P_0$ following directions generated by RNG till an absorbing point is hit. A point
is said to be absorbing if it is a boundary point in the elliptic case, while in the parabolic case, it
is absorbing if it is either a boundary point or a point reached at a certain specified time.

![Figure 1](image_url)

Let $r(P_0) = 2(A - B + C) + h(E + D)$, where $A, B, C, D, E$ and $F$ are evaluated at $(x, y)$,
the coordinates of the point $P_0$. Let also $W_i$ be a random walk starting at $P_0$ and ending at a
boundary point $Q_i$, and

$$Z(W_i) = \phi(Q_i) + h \sum_{P_j \in W_i} \frac{F(P_j)}{r(P_j)}$$  \hspace{1cm} (5)

The Monte Carlo solution of the elliptic equation (1) at point $P_0$, subject to condition (2),
consists of generating a number of random walks $W_1, W_2, \ldots, W_N$, all starting at $P_0$ and ending
at $Q_1, Q_2, \ldots, Q_N$, respectively. Afterwards, $Z(W_i)$ is evaluated for all $i = 1, 2, \ldots, N$.

Finally, $U(P_0)$ is approximated by

$$
\theta = \frac{1}{N} \sum_{i=1}^{N} Z(W_i)
$$

(6)

$Z(W_i)$ is called the primary estimator of $U(P_0)$, and $\theta$ the secondary estimator. For the proof that this method yields a good approximation of $U$, see [1, num].

For the parabolic case, where $U$ and the coefficients of (1) are time dependent, the time scale is discretized into equal units of length $k$ (i.e., $t_s = sk$, $s \geq 0$), and $U(P_0)$, $\phi(P_0)$, $A_s$, $B_s$, $C_s$, $D_s$, $E_s$, $F_s$ and $r_s(P_0)$ at time $t_s$ are denoted $U_s(P_0)$, $\phi_s(P_0)$, $A_s$, $B_s$, $C_s$, $D_s$, $E_s$, $F_s$ and $r_s(P_0)$, respectively.

A random walk $W$ in this case is constructed as before except that $W$ is started at $P_0$ at time $t_s = sk$, and after each step (following a new direction), the time is decreased by one unit. The walk $W$ is finished if either a boundary point is reached or the time runs out (after $s$ steps), whichever comes first.

In this case, the primary estimator $Z(W_i)$ of $U_s(P_0)$ is:

$$
Z(W_i) = V_s(Q_i) + h^2 \sum_{j=0}^{s-a} \frac{F_{s-j}(P_j)}{r_{s-j}(P_j)}
$$

(7)

where $Q_i$ is the last point of $W$ reached at time $t_s = ak$, and

$$
V_s(Q_i) = \begin{cases} 
\phi_s(Q_i) & \text{if } Q_i \in \gamma \text{ and } a \geq 0 \\
g(Q_i) & \text{if } a = 0 \text{ (i.e., } Q_i \text{ is reached at time } 0, \text{ and may be a non-boundary point)}
\end{cases}
$$

The Monte Carlo solution of the parabolic equation (1) at point $P_0$, at time $t_s$, subject to the conditions (3) and (4), consists of generating $W_1, W_2, \ldots, W_N$, evaluating the $Z(W_i)$'s and averaging them as in the elliptic case.

It can be clearly seen that the random walks $W_1, W_2, \ldots, W_N$ are independent, and that $Z(W_1), Z(W_2), \ldots, Z(W_N)$ can be computed independently (i.e., in parallel). This inter-walk parallelism has been studied in [2], [3], [11]. It is also clear to see that $U$ can be computed at different points independently and thus concurrently.

The third area for parallelism is the construction of each random $W$ and the computation of $Z(W)$ along with it. We refer to this process as the random walk computation (RWC). At first glance, the construction of a random walk seems inherently sequential since a current point has to
be known before the succeeding point is found using a random direction generator. However, this paper will parallelize the construction of a random walk.

§3. Parallel Construction of Random Walks

To parallelize the random walk construction we first need a number of independent random number (i.e., direction) generators that generate a sequence of random numbers simultaneously. Assume we have \( n - 1 \) independent random number generators running on \( n - 1 \) processors (the choice of \( n - 1 \) as opposed to \( n \) will be justified later). The problem can be stated as follows:

Given a grid, a point \( P_0 \) in the grid, and a sequence of \( n - 1 \) random directions \( d_1, d_2, \ldots, d_{n-1} \) generated by the \( n - 1 \) random number generators, construct in parallel the random walk that starts at \( P_0 \) and moves away following the directions \( d_1, d_2, \ldots, d_{n-1} \), consecutively. That is, the walk is a sequence of points \( P_0, P_1, P_2, \ldots, P_{n-1} \) such that \( P_i \) is the \( d_i \)-th neighbor of \( P_{i-1} \) for \( i = 1, 2, \ldots, n - 1 \).

For simplicity, assume first that the grid is an \( m \times k \) rectangular grid labeled in such a way that the grid point \((i, j)\) is in the \( i \)-th row and \( j \)-th column, where \((0, 0)\) is in the bottom leftmost point. The case where the grid is not rectangular will be handled later.

A move along a direction \( d \) can be viewed as a translation \( T_A \) for some vector \( A = (a, b) \) such that \( T_A(i, j) = (i, j) + A = (i + a, j + b) \). The translations corresponding to the five directions are:

\( T_{(0,1)} \) for the move to the east direction, \( T_{(0,-1)} \) for the move to the west direction, \( T_{(1,0)} \) for the move to the north direction, \( T_{(-1,0)} \) for the move to the south direction, and \( T_{(-1,1)} \) for the move to the southeast direction.

Given an initial grid point \( P_0 \) and \( n - 1 \) random directions \( d_1, d_2, \ldots, d_{n-1} \), the corresponding translations \( T_{A_1}, T_{A_2}, \ldots, T_{A_{n-1}} \) can be simply found and the points \( P_1, P_2, \ldots, P_{n-1} \) are derived as follows: \( P_1 = T_{A_1}(P_0) = P_0 + A_1, P_2 = T_{A_2}(P_1) = T_{A_2}T_{A_1}(P_0) = P_0 + A_1 + A_2 \), and in general, \( P_i = T_{A_i}(P_{i-1}) = T_{A_i} \cdots T_{A_2}T_{A_1}(P_0) = P_0 + A_1 + A_2 + \cdots + A_i \), for \( i = 1, 2, \ldots, n - 1 \). Finding these points is now a standard parallel prefix computation [7] [8] [9] which can be solved in \( O(\log n) \) time. However, further work has to be done to check boundary crossing. A point \( P_i = (p, q) \) is a boundary point simply if \( p = 0, p = m - 1, q = 0 \) or \( q = k - 1 \), but what is required is to determine in a parallel setting, the first boundary point and to discard all the remaining points after it. If there is no boundary crossing and \( P_n \) is not a boundary point, the algorithm is repeated. The
details of this algorithms are elaborated next.

The algorithm makes a heavy use of the procedure SCAN which does the prefix computation on $n$ processors in parallel taking $O(\log n)$ computation time, and, if run on a hypercube of $n$ processors, it takes $O(\log n)$ communication time. A similar procedure is implemented on the Connection Machine. Subsection 3.1 will present the procedure SCAN. Subsection 3.2 will describe the parallel (RWC) for the elliptic PDE's on rectangular grids. The case of non-rectangular grids is shown to be a slight variation of the rectangular case and is handled in Subsection 3.3. Afterwards, the necessary modifications to handle parabolic PDE's are discussed in Subsection 3.4.

3.1 The SCAN Algorithm

The procedure SCAN performs the parallel prefix computation for any associative operator. Specifically, the prefix problem is to compute the values of $X_0, X_0 \circ X_1, X_0 \circ X_1 \circ X_2, \ldots, X_0 \circ X_1 \circ \cdots \circ X_{n-1}$, given the values $X_0, X_1, \ldots, X_{n-1}$. The operator $\circ$ is any arbitrary associative binary operation such as scalar addition ($+$), vector addition ($+$), the minimum operator (MIN) and so on. The integer $n$ can be assumed to be a power of 2. The parallel prefix problem is to compute the values above in parallel.

The parallel prefix problem has received much attention [7], [8], [9], and various VLSI circuit implementations have been proposed [9]. [10], [11]. We will present in this paper an optimal parallel algorithm for prefix computation. The algorithm communication structure will be shown to ideally fit on a hypercube network of $n$ nodes.

The algorithm for SCAN is best explained first in a recursive way. Denote by $X_{i,j}$ the value of $X_0 \circ X_1 \circ \cdots \circ X_j$. Initially processor $pe_i$ has $X_i$. At the end of the algorithm $pe_i$ will have the value $X_0$, and $X_0 \circ X_{n-1}$. To understand the basis step of the recursive algorithm, assume there are 2 processors only, that is, $n = 2$. Then the algorithm proceeds as follows. $pe_0$ sends $X_0$ to $pe_1$ and $pe_1$ sends $X_1$ to $pe_0$. Afterwards, every $pe$ computes $X_{0,1} := X_0 \circ X_1$. Thus, $pe_0$ has now $X_0$ (which is $X_0$), $pe_1$ has $X_{0,1}$, and both $pe$'s have $X_{0,1}$.

To understand the recursive step, assume SCAN has been performed by the first half-interval of processors $pe_0, pe_1, \ldots, pe_{\frac{n}{2}-1}$ on the data $X_0, X_1, \ldots, X_{\frac{n}{2}-1}$, and also by the second half-interval of processors $pe_{\frac{n}{2}}, pe_{\frac{n}{2}+1}, \ldots, pe_{n-1}$ on the data $X_{\frac{n}{2}}, X_{\frac{n}{2}+1}, \ldots, X_{n-1}$. Assume also that the effect of this two SCAN's is that for every $i = 0, 1, \ldots, \frac{n}{2} - 1$, $pe_i$ has the value $X_{0,i}$ and
the value $X_{0:q-1}$, and that for every $i = \frac{q}{2}, \frac{q}{2} + 1, \ldots, n - 1$, $pe_i$ has the value $X_{q:i}$, and the value $X_{q:n-1}$. After the recursive step, every processor $pe_i$, for $i = 0, 1, \ldots, n - 1$, will have the values $X_{0:i}$ and $X_{0:n-1}$. This step is accomplished as follows. Every processor $pe_i$ in the first half-interval sends the value $X_{0:q-1}$ to $pe_{i+\frac{q}{2}}$ in the second half-interval. Similarly, every processor $pe_{i+\frac{q}{2}}$ in the second half-interval sends the value $X_{q:n-1}$ to $pe_i$ in the first half-interval. Afterwards, every $pe_i$ in the second half-interval computes $X_{0:i} := X_{0:q-1} \circ X_{q:i}$. Finally, every $pe_i$ in both half-intervals computes $X_{0:n-1} := X_{0:q-1} \circ X_{q:n-1}$. By this recursive step, every $pe_i$ has $X_{0:i}$ and $X_{0:n-1}$.

This algorithm can be implemented nonrecursively in $\log n$ stages as follows. In the first stage, every pair of processors $pe_{2i}$ and $pe_{2i+1}$ do the same on their respective data as is done in the basis step of the recursive algorithm explained above. At stage $i$, the $n$ processors are divided into $\frac{n}{2^i}$ independent intervals $(I_j)_{0 \leq j \leq \frac{n}{2^i} - 1}$, where $I_j = [j2^i, (j + 1)2^i - 1] = (j2^i, j2^i + 1, j2^i + 2, \ldots, (j + 1)2^i - 1]$. Each interval $I_j$ is in turn divided into two half-intervals of processors $[j2^i, j2^i + 2^{i-1} - 1]$ and $[j2^i + 2^{i-1}, (j + 1)2^i - 1]$ such that every $pe_i$ in the first half-interval has the values $X_{j2^i:i}$ and $X_{j2^i:j2^i+2^{i-1}-1}$, and every $pe_i$ in the second half-interval has the values $X_{j2^i+2^{i-1}:i}$ and $X_{j2^i+2^{i-1}:(j+1)2^i-1}$. In stage $i$, the processors in these two half-intervals perform the same job on their data as the two half-intervals in the recursive step in the last paragraph. The details of this job are presented in the first inner for-loop in the procedure Stage(i) below. This procedure implements the $i$-th stage just explained.

To fully understand the working of Stage(i), the semantics of three special parallel language constructs in Stage(i) need to be specified. The first is of the form:

for $j = m$ to $k$ pardo

proc $j$;

endfor

which means that the processes proc$_m$, proc$_{m+1}$, $\ldots$, proc$_k$ run simultaneously.

The second is of the form: $pe_i$ does: $S$; which means that processors $pe_i$ executes the statement $S$. The third is of the form Send ($pe_i$, $a$, $pe_s$); which means that processor $pe_i$ sends the data value $a$ to processor $pe_s$.

Procedure Stage(i);

begin

560
for $j = 0$ to $\frac{n}{2} - 1$ pardo /*$j$ denotes the interval $I_j = [j2^i, (j + 1)2^i - 1]$ of pe's*/
for $l = j2^i$ to $j2^i + 2^{i-1} - 1$ pardo /*$l$ ranges over the first half of $I_j$*/
    Send $(pe_i, X_{j2^i: j2^i + 2^{i-1} - 1}, pe_{i+2^{i-1}})$;
    Send $(pe_{i+2^{i-1}}, X_{j2^i + 2^{i-1}, (j+1)2^i - 1}, pe_i)$;
    pe_{i+2^{i-1}} does: $X_{j2^i: j2^i + 2^{i-1} - 1} := X_{j2^i: j2^i + 2^{i-1} - 1} \circ X_{j2^i + 2^{i-1}, (j+1)2^i - 1}$;
endfor
for $l = j2^i$ to $(j + 1)2^i - 1$ pardo
    pe_i does: $X_{j2^i: (j+1)2^i - 1} := X_{j2^i: j2^i + 2^{i-1} - 1} \circ X_{j2^i + 2^{i-1}, (j+1)2^i - 1}$;
endfor
end

The Algorithm for the $i$-th Stage of SCAN

The full algorithm for SCAN is a simple sequential for-loop executing stage 1, stage 2, ... , stage $\log n$, as presented below.

Algorithm SCAN($X(0..n - 1)$);
begin
    for $i = 1$ to $\log n$ do
        Stage($i$);
    endfor
end

The Algorithm for SCAN

Communication and Complexity analysis of SCAN

By a simple inspection of the procedure Stage, we observe that communication occurs between processor $pe_i$ and $pe_{i+2^{i-1}}$ for various values of $l$ and $i$ such that $i = 1, 2, ..., \log n$ and $j2^i \leq l \leq j2^i + 2^{i-1} - 1$. When $l$ is expressed as a binary number $l_{n-1}...l_1 l_0$, it can be seen that $l_{i-1}$ is equal to 0 and that $l + 2^{i-1}$ has the same binary representation as $l$ except that bit $l_{i-1}$ is complemented. That is, $l$ and $l + 2^{i-1}$ differ in only one bit. Consequently, if SCAN is run on a hypercube system of $n$ processors, every two processors that need to communicate will have a direct link between them. In other terms, the hypercube structure is an ideal structure for SCAN. As for the time complexity,
the body of the two inner for-loops in Stage takes constant time. Since all the for-loops in Stage are parallel loops, Stage(i) takes constant time. It follows that SCAN takes $O(\log n)$ time. Finally, as explained earlier, every processor needs to store two values only at any stage. Thus the space complexity is optimal.

Note that when the operator $\circ$ is scalar addition, we refer to SCAN as ADD-SCAN. When $\circ$ is vector addition, SCAN is referred to as VADD-SCAN, and when $\circ$ is the minimum operator MIN, we refer to SCAN as MIN-SCAN.

3.2 Parallel RWC for the Elliptic PDE’s

Given an initial grid point $P_0$ and $n-1$ directions corresponding to the translations $T_{A_1}, T_{A_2}, \ldots, T_{A_{n-1}}$, the points $P_0, P_1, \ldots, P_{n-1}$ of the random walk are found by calling VADD-SCAN($A(0..n-1)$), where $A(0) = P_0$ and $A(i) = A_i$ for $i = 1,2,\ldots,n-1$. Since $P_i = P_0 + A_1 + A_2 + \ldots + A_i$, we conclude that $P_i = A_{0,i}$ and is hence computed by $pe_i$. At this point, the boundary checking test has to be performed.

Assume that the coordinates of $P_i$ are $(a_i, b_i)$, or equivalently, that $A_{0,i} = (a_i, b_i)$. Each $pe_i$ will check if the point $P_i$ is a boundary point and compute a certain flag $f_i$, as follows:

```
Check-Boundary($P_i$) /* done by $pe_i$ */
begin
    if ($a_i = 0$ or $a_i = m - 1$) or ($b_i = 0$ or $b_i = k - 1$) then
        /* $P_i$ is on boundary*/
        $f_i := i$
    else
        $f_i := 2n$; /* or any number > $n$*/
end
```

To determine the first boundary point so that all succeeding points are discarded, we determine the minimum of all the flags $f_0, f_1, \ldots, f_{n-1}$. This can be accomplished by executing MIN-SCAN($f(0..n-1)$). The minimum $min$ is $f_{0:n-1}$ which is available at every $pe_i$ at the end of MIN-SCAN. If all the points are within boundary, then each $f_i = 2n$ and hence the minimum is $2n$. If there is a boundary point, assume that $P_i$ is the first boundary point. Therefore, $f_i = l$, and for $i < l$, $f_i = 2n > f_i$. For $i > l$, $f_i$ is either $i$ or $2n$ based on whether $P_i$ is a boundary point or
not; in either case \( f_i = l < f_i \). Thus, \( f_i \) is the minimum of all the \( f_i \)'s, and is equal to \( l \). It follows that \( \min \) is equal to \( 2n \) if all points are within boundary, and if there is a boundary point, \( \min \) is equal to the index of the first boundary point (which is \( < 2n \)), that is, \( P_{\min} \) is the first boundary point.

After the boundary checking and the computation of the minimum \( \min \) of the flags \( f_i \)'s, each \( pe_i \) checks if \( \min = 2n \) (recall that every \( pe_i \) has the \( \min \) after MIN-SCAN). If \( pe_j \) finds \( \min = 2n \) or \( j \leq \min \), then \( pe_j \) computes the value \( z_j = \frac{F(P_j)}{n(P_j)} h^2 \) because the point \( P_j \) is on the walk. Otherwise, the point \( P_j \) is after the first boundary point, in which case \( pe_j \) sets \( z_j \) to 0.

Now if \( \min \neq 2n \), then \( P_{\min} \) is the first boundary point (i.e., the endpoint \( Q \) of the random walk \( W \) just generated). In this case, \( pe_{\min} \) sets \( x_{\min} \) to \( x_{\min} := x_{\min} + \phi(P_{\min}) \). Afterwards, the processors \( pe_0, pe_1, \ldots, pe_{n-1} \) sum their \( z_i \)'s to form \( Z(W) \). This is accomplished by executing ADD-SCAN\((z(0..n-1))\).

On the other hand, if \( \min = 2n \), then no boundary point has been reached yet. The sum \( x_0 + x_1 + \ldots + x_{n-1} \) is computed using ADD-SCAN\((z(0..n-1))\), and the stored in a temporary variable \( Z \) in \( pe_0 \). Afterwards, \( pe_0 \) sets the points \( P_0 \) to \( P_{n-1} \) (i.e., \( P_0 := P_{n-1} \)) while all the other \( pe \)'s clear all their variable, and the whole process of generating \( n-1 \) random directions and finding new points is repeated. The "\( Z \)-value" of every new set of points is computed and added to the old \( Z \) variable in \( pe_0 \). The process is repeated till a boundary point is reached.

The execution time of the overall algorithm for parallel RWC is \( O(n \log n) \) on a hypercube architecture of \( n \) processors, where \( L \) is the length of the random walk. This is so because each iteration of the algorithm corresponding to a portion of \( n-1 \) points of the path take \( O(\log n) \) time.

This brings an end to the parallel RWC algorithm for rectangular grids and elliptic PDE's. In the next two subsections, non-rectangular regions are handled and then the modifications needed to handle the parabolic PDE's are presented.

### 3.3 Handling Non-Rectangular Regions

For the case of non-rectangular regions, the region is embedded in the smallest rectangular \( m \times k \) grid where the boundary lines of the grid are tangent to the region. The two points of intersection between the region boundary and row \( p \) of the grid are recorded for each \( p \). The western intersection point takes the label of the grid point immediately to its west (denoted \( (p, W(p)) \)), and the eastern
intersection point takes the label of the grid point immediately to its east (denoted \( p, E(p) \)). Similarly, the two points of intersection between the region boundary and column \( q \) of the grid are recorded for each \( q \). The northern intersection point takes the label of the grid point immediately to its north (denoted \( N(q), q \)), and the southern intersection point takes the label of the grid point immediately to its south denoted \( S(q), q \). The random walk generation is the same as in the rectangular case except for boundary checking. A point \( P = (p, q) \) is a boundary point if \( p = 0 \) or \( S(q) \), \( p = m - 1 \) or \( N(q) \), \( q = 0 \) or \( W(p) \), or \( q = k - 1 \) or \( E(p) \). Thus the Check-Boundary procedure becomes:

\[
\text{Check-Boundary}(P) \quad /\ast \text{ done by } \text{pe}_i, \quad P = (a_i, b)_i /\ast \\
\text{begin} \\
\quad \text{if } (a = 0 \text{ or } a = S(b)) \text{ or } a = m - 1 \text{ or } a = N(b) \\
\quad \quad \text{ or } (b = 0 \text{ or } b = W(a)) \text{ or } b = k - 1 \text{ or } b = E(a) \text{ then} \\
\quad \quad \quad /\ast P, \text{ is on boundary} /\ast \\
\quad \quad \quad f_i := i; \\
\quad \text{else} \\
\quad \quad f_i := 2n; /\ast \text{ or any number } > n /\ast \\
\text{end} \\
\]

To be able to execute this procedure, every \text{pe} has to store \( W(p) \), \( E(p) \), \( N(q) \), and \( S(q) \) for all \( 0 \leq p \leq m - 1 \) and \( 0 \leq q \leq k - 1 \).

Thus the parallel RWC algorithm keeps its simplicity and speed, requiring only some additional storage for the intersection points between the region boundary and the grid.

3.4 Parallel RWC for Parabolic PDE's

The parallel RWC algorithm for the parabolic case is very similar to the algorithm for the elliptic case. The only difference is the definition of absorbing points and the subsequent change needed to detect boundary points and the slight modification of \( Z(W) \).

At the outset of the algorithm, every \text{pe}, has a counter \( T \) (for time) initialized to \( s \). The algorithm finds the \( n - 1 \) points first in the same way as in the elliptic case. It also performs boundary checking as before. If a boundary point has been detected, say \( P_{\text{min}} \), and if \( T - \text{min} \geq 0 \), then \( P_{\text{min}} \) is an absorbing point, and hence every \text{pe}_j, for \( j \leq \text{min} \), computes \( x_j := \frac{E_{T-1}(P_j)}{E_{T-1}(P_j)} h^2 \).
and then $p_{e_{\text{min}}}$ sets $x_{\text{min}}$ to $x_{\text{min}} + q_{e_{\text{min}}}(P_{\text{min}})$ as required to compute $Z(W)$ of equation (7). All the remaining $pe$'s set their $x$'s to 0. $Z(W)$ is then computed by executing ADD-SCAN as in the elliptic case.

If on the other hand, $T - \min < 0$ (i.e., run out of time), then the absorbing point is $P_T$. In this case, only the $pe_j$'s where $j \leq T$ compute $x_j := \frac{F_{T-j}(P_j)}{F_{T-j}(P_T)}h^j$, while all the remaining $pe$'s set their $x$'s to 0. The remaining part to compute $Z(W)$ is the same as in the previous paragraph.

If $\min = 2n$ and no grid boundary point is reached, and if $T \leq n - 1$, then $P_T$ is an absorbing point and the algorithm does as in the preceding paragraph. However, if $T > n - 1$, then no absorbing point has been reached. In this case, the same computations are done as in the elliptic case (to accumulate $Z$), but before we repeat the algorithm with a new set of $n - 1$ random directions, the counter $T$ in each $pe$ updated: $T := T - (n - 1)$. Afterwards, the algorithm is repeated till an absorbing point is reached.

As can be seen, the additional computations needed for the parabolic case takes constant time. Consequently, the overall time for the parallel RWC for the elliptic or parabolic PDE's is $O(\frac{L}{n} \log n)$, whether the region is a rectangular grid or not, where $L$ is the length of the random walk, and $n$ is the number of processors.

§4. Conclusions

We have presented in this paper a parallel algorithm for the construction of random walks in the Monte Carlo solution of elliptic and parabolic partial differential equations. The algorithm was shown to ideally fit on a hypercube structure. The algorithm is optimal in time and space when the region is a rectangular grid. It is also optimal in time when the region is irregular. In the latter case, a certain amount of space is needed at every processor, and it is open whether it can be done in constant space per processor.

This parallel generation of random walks offers great speedup in the solution of partial differential equations. It reduces the time of random walk construction from linear to logarithmic time in the length of the random walk.

Finally, the parallel generation algorithm presented here is easily generalizable to grids of any dimensions and can have applications in other areas. Future work will investigate parallel random walk generation in grids of geometries different from rectangular grids.
§5. References


